NOTES ON LATTICES

By L. M. Blumenthal and D. O. Ellis

1. **Introduction.** Let L be a lattice with inclusion relation (in the wide sense) $\subset$, meet ab and join $a + b$. When L is assumed normed, the norm of an element x is denoted by $|x|$. Distance is introduced into a normed lattice by attaching to each pair of elements a, b the number $(a, b) = |a + b| - |ab|$. The resulting space is a metric space, the *associated metric space* of L, denoted by $D(L)$.

Relations between lattice properties of a normed lattice L and metric properties of the associated metric space $D(L)$ are of considerable interest. A study of such relations was begun by Glivenko in 1936 [2]. He showed, for example, that if $a, b, c \in D(L)$, then b is metrically between a and b (that is, $(a, b) + (b, c) = (a, c)$) if and only if

$$\text{(G)} \qquad ab + bc = b = (a + b)(b + c).$$

(It is convenient when studying betweenness in lattice theory *not* to demand that the points be pairwise distinct, as is usually done in a purely metric study of that notion.) We shall denote that b is metrically between a and c by writing abc.

This paper is part of such a program. Glivenko's lattice characterization (G) of metric betweenness is formally self-dual. (Another self-dual necessary and sufficient condition for metric betweenness in normed lattices (also due to Glivenko) is $a(b + c) \subset b \subset a + bc$.) Two lattice characterizations of metric betweenness are obtained in §2, neither of which is formally self-dual. That section deals also with the role of pseudo-linear quadruples in lattice theory, presenting sufficient, necessary, and necessary and sufficient conditions, in terms of the lattice operations, that four distinct points of $D(L)$ form a pseudo-linear quadruple.

The importance of pseudo-linear quadruples in lattice theory is evidenced by a theorem of §3 which proves that $D(L)$ is congruent with a subset of Hilbert space if and only if pseudo-linear quadruples are absent—in which case, the lattice is congruently imbeddable in the straight line.

A one-to-one mapping of one normed lattice onto another has property (M), (N), or (D) according as it preserves meets, norms (modulo a constant), or distances, respectively. It is shown in §4 that any two of these properties imply the third.

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