A NOTE ON THE ISOMETRIC CORRESPONDENCE OF RIEMANN SPACES

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In a Riemann space (in the small) consider the set of all absolute scalar invariants. If p $(\leq n)$ of these are functionally independent and if the other invariants are functionally dependent on them, the space is said to be of category p. For Riemann spaces of the same category, Prof. Thomas [2] has set up a set of scalar relationships which imply that the spaces are isometric. The problem was then solved for n-dimensional spaces of category n, and for two-dimensional spaces of categories 2, 1, and 0. He left open the problem of the extension of his results to Riemann spaces R_n when $n > 2$.

It is a purpose of this note to solve the problem he left for Riemann spaces R_n of category $n - 1$.

Let $I_1, I_2, \cdots, I_{n-1}$ be a fundamental set of scalar invariants of a space R of category $n - 1$; we form two sets of absolute scalar invariants I_{jk} and J_i where

$$I_{jk} = g^{\alpha\beta} I_{j,\alpha} I_{k,\beta}, \qquad J_i = g^{\alpha\beta} I_{i,\alpha\beta}. \tag{1}$$

(Here and hereafter we use italic letters running from 1 to $n - 1$, and Greek letters from 1 to n.) The quantities $I_{k,\alpha}$ are partial derivatives with respect to x^α, and $I_{k,\alpha\beta}$ are the components of the second covariant derivatives of I_k for k fixed.

Before going to the fundamental theorem we have a lemma which will be of essential use.

Lemma. *A Riemann space R_n of category $n - 1$ having fundamental scalar invariants $I_1, I_2, \cdots, I_{n-1}$ admits coordinates $y^1, y^2, \cdots, y^n$ (covering any point P of R_n) for which $I_1 = y^1$, $I_2 = y^2$, $\cdots$, $I_{n-1} = y^{n-1}$ and the line-element has the form*

$$dS^2 = p_{ij}(y^1, y^2, \cdots, y^{n-1})\, dy^i\, dy^j + p_{nn}(y^1, y^2, \cdots, y^{n-1})(dy^n)^2. \tag{2}$$

Proof. Without loss of generality we can suppose

$$| \partial I_i / \partial x^j | \neq 0 \qquad (i, j = 1, \cdots, n - 1) \tag{3}$$

at any point P of R_n where $x^1, x^2, \cdots, x^n$ are the coordinates of a suitably chosen system. Then $I_i(x^1, \cdots, x^n) = \eta^i$ $(i = 1, \cdots, n - 1)$, $x^n = \eta^n$ defines a non-singular transformation of a neighborhood of P. Let $g_{\alpha\beta}(x) \rightarrow \theta_{\alpha\beta}(\eta)$ be this transformation. Now consider the system of $n - 1$ equations

$$\theta^{ij} \partial v / \partial \eta^j = 0. \tag{4}$$

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