THETA SERIES IN THE GAUSSIAN FIELD

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1. **Introauction.** We consider in this paper the function

$$f_r(\tau) = \sum_{\mu} \mu^r e^{\pi i \tau \mu \mu'} \qquad (r \text{ integer} > 0) \tag{1.1}$$

where μ runs through all the integers in the Gaussian field. Glaisher [2] has studied in detail the functions $f_4(\tau)$ and $f_8(\tau)$ and has expressed them in terms of Jacobian elliptic functions. We derive a formula expressing $f_r(\tau)$ in terms of the known theta functions. The methods used depend on the fact that $f_r(\tau)$ is a combination of two theta series of a type defined by Hecke, and can therefore be shown to be a modular form of dimension $-(r + 1)$ associated with a subgroup of the full modular group.

Hecke defined [4; 696]

$$\vartheta_r(\tau;\, \rho,\, \mathfrak{a},\, D^{\frac{1}{2}}) = \sum_{\mu} \mu^{r-1} \exp\,(2\pi i \tau \mu \mu'/A \mid D \mid) \qquad (\mu \equiv \rho \quad (\mathfrak{a} D^{\frac{1}{2}})),$$

where $\mathfrak{a}$ is an integral ideal in an imaginary quadratic field $K(D^{\frac{1}{2}})$ with discriminant D, ρ is a number in $\mathfrak{a}$, $A = \mid N(\mathfrak{a}) \mid$, r is a positive integer, and τ is a complex number with positive imaginary part.

For our purposes here we may consider these series in the Gaussian field, and take $\mathfrak{a} = (1)$. Changing the notation slightly, we discuss in §2 the functions

$$\vartheta_r(\tau,\, \rho,\, Q) = \sum_{\mu \equiv \rho \ (2Q)} \mu^r \exp\,(2\pi i \tau \mu \mu'/4Q) \qquad (r > 0) \tag{1.2}$$

and derive transformation formulae showing their behavior when $Q = 1$ under arbitrary modular substitutions. Since $\vartheta_r(\tau,\, \rho,\, 1) \equiv 0$ if r is odd, only even values of r are considered. As a result of these formulae it is seen that $\omega_2^{-(r+1)} \vartheta_r(\omega_1/\omega_2\,,\, \rho,\, 1)$ is a modular form of dimension $-(r + 1)$ and level 4, or, expressed in non-homogeneous form, that

$$\vartheta_r\left(\frac{a\tau + b}{c\tau + d},\, \rho,\, 1\right) = (c\tau + d)^{r+1} \vartheta_r(\tau,\, \rho,\, 1)$$

for all modular substitutions $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ modulo 4.

The transformation formulae obtained of course show the behavior under arbitrary modular substitutions of any linear combination of these four theta series. For convenience of reference, these results are appended at the end of

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