POLYNOMIALS WITH ZEROS ON A RECTIFIABLE JORDAN CURVE

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1. **Introduction.** The principal object of this paper is to prove the following theorem.

Theorem I. *Let D be a bounded simply-connected domain in the z-plane, bounded by a rectifiable Jordan curve Γ. Let $f(z)$ be holomorphic and never zero in D. There exists a sequence of polynomials $P_n(z)$, $n = 1, 2, \cdots$, such that each $P_n(z)$ has all of its zeros on Γ, and such that $P_n(z) \to f(z)$ as $n \to \infty$, uniformly in any closed subset of D. The subscript n is not intended to imply anything about the degree of $P_n(z)$.*

There are two obvious aspects of this theorem that are of interest: firstly, it illustrates the possible latitude in the choice of approximating polynomials (see §7, Theorem IV); and, secondly, it may be thought of as complementary to the theorem of Jentzsch [3], which states that the closure of the set of all zeros of the partial sums of a Taylor series contains the circumference of the circle of convergence.

In §2 representations are chosen for $f(z)$ and $P_n(z)$, the similarity of which is the basis for the proof; the actual sequence of polynomials is defined in §3, and the convergence to $f(z)$ is demonstrated in §4, under the assumption that Γ is analytic (Theorem II). This special case is proved first, since it is used in proving the general case; also in this case $P_n(z)$ may be chosen of degree n in such a way that the convergence is geometric in any closed subset of D, provided that $f(z)$ is holomorphic in $D + \Gamma$. In §5 Theorem I is proved, while §6 is devoted to an extension to multiply-connected domains (Theorem III).

In connection with the question, how much may one restrict the zeros of approximating polynomials, Theorem I is completed (Theorem IV) by showing that the zeros may be placed on any rectifiable Jordan curve which surrounds the domain in which $f(z)$ is to be approximated. The distribution of the zeros along Γ is the subject of §8 (Theorem V); in particular, it is shown that these must be everywhere dense on Γ; and hence the zeros cannot be essentially more restricted.

The simplest possible examples of Theorem I are the following: Let Γ be the circle $|z| = 1$. If $f(z) \equiv 1$, we may take $P_n(z) = 1 - z^n$. If $f(z) = (z - e^{ib})^{-1}$, b real, we may take

$$P_n(z) = \frac{1 - z^n e^{-nib}}{z - e^{ib}} = -e^{-nib} \prod_{k=1}^{n-1} (z - e^{ib+2\pi ik/n}).$$

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