DISTORTION OF LENGTH IN CONFORMAL MAPPING

By P. R. Garabedian

1. The auxiliary problem. We solve in this paper a minimum length problem in the theory of functions by investigating a closely related problem of the type of Schwarz's lemma. The related problem is attacked by a new and powerful method of variation of branch points of extremal Riemann surfaces. This simple variation will be seen to have wide applications throughout conformal mapping, and it leads readily to many dual pairs of extremal problems by the introduction of an associated quadratic differential. The technique we shall use has been in large part introduced by Garabedian and Schiffer [3], while the idea of varying branch points is allied to the work of Ahlfors [1]. The paper is more or less a natural continuation of the author's thesis [2].

We consider analytic functions $F(z)$ in a finite domain D bounded by n analytic curves $C_1, \cdots, C_n$, and we require that for each individual F there exist constants $b_1, \cdots, b_n$ such that

$$\overline{\lim_{z \to C_\mu}} \mid F(z) - b_\mu \mid \leq 1 \qquad (\mu = 1, \cdots, n). \tag{1}$$

We denote by Ω_{z_0} the subclass of these functions $F(z)$ with

$$F(z_0) = 0, \qquad F'(z_0) \geq 0, \tag{2}$$

for any $z_0 \varepsilon D$. We denote further by $\Omega_{z_0}^m$ the subclass of functions $F \varepsilon \Omega_{z_0}$ which are at most m-valued in D, which are analytic on the boundary C of D, and which satisfy conditions

$$\mid F(z) - b_\mu \mid \equiv 1 \qquad (z \varepsilon C_\mu ; \mu = 1, \cdots, n). \tag{3}$$

Here the b_μ denote constants which depend on F. An application of Schwarz's principle of reflection shows that the family $\Omega_{z_0}^m$ is normal and compact. $\Omega_{z_0}^m$ is not empty for $m \geq n$, since there exist maps $F \varepsilon \Omega_{z_0}^n$ satisfying (3) with $b_1 = b_2 = \cdots = b_n = 0$. (See, for example, [2].)

There exists in $\Omega_{z_0}^m$ an extremal function $F_0(z)$ with

$$F_0'(z_0) = \max F'(z_0) \qquad (F \varepsilon \Omega_{z_0}^m). \tag{4}$$

Since $\Omega_{z_0}^n$ has elements with a non-vanishing derivative at z_0 (see [2]), we see that $F_0'(z_0) \neq 0$, and it follows that $F_0(z)$ is not a constant. F_0 maps D upon a Riemann surface S over the ζ-plane whose boundary components lie over non-concentric unit circles defined by relations of the form (3). S has certain branch points $a_1, \cdots, a_k$ which we propose to vary in such a manner that the conformal type of S remains invariant. This motion of branch points will define a new map $F^*(z)$ of D upon the varied surface S^*, and by comparing

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