MULTIPLICATIVE SEMIGROUPS OF CONTINUOUS FUNCTIONS

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1. **Introduction.** We illustrate the contents of this paper with an example. Let C_0 denote the set of all convergent sequences of real numbers, $C_0 = [(a_1, a_2, \cdots, a_n, \cdots)]$. We regard C_0 as a semigroup in which if $a = (a_1, a_2, \cdots)$, $b = (b_1, b_2, \cdots)$ then $a \cdot b = (a_1b_1, a_2b_2, \cdots)$. If X_0 is a space consisting of a denumerable set of points $P_1, P_2, \cdots$ converging to a single point P_0, then C_0 may be regarded as the set of continuous functions $C_0 = C(X_0)$ where $f \varepsilon C(X_0)$ is associated with $f(P_1), f(P_2), \cdots$. The set $C(X)$ of continuous real functions defined on a bicompact Hausdorff space X may always be regarded as a semigroup in which multiplication is defined pointwise, that is, for $x \varepsilon X$ and $f, g \varepsilon C(X)$ we define $f \cdot g(x) = f(x)g(x)$.

Suppose that the semigroup C_0 is isomorphic to some semigroup C_1, where C_1 is known to be the semigroup of continuous functions on a bicompact space X_1. Then it will follow that X_1 must also be a convergent sequence of points, so that the isomorphism can really be considered to be an automorphism. This is a consequence of the fact that *if $C(X)$ and $C(X')$ are isomorphic semigroups, then the bicompact spaces X and X' are homeomorphic.* (This was conjectured by Ky Fan during a conversation with the author.)

What are the different kinds of automorphisms of C_0? We first name three simple types. (α) If $H_0 : (1, \cdots, n, \cdots) \rightarrow (i_1, i_2, \cdots, i_n, \cdots)$ is a permutation of the positive integers, then $H_0^* : C_0 \rightarrow C_0$ defined by $(a_1, a_2, \cdots) \rightarrow (a_{i_1}, a_{i_2}, \cdots)$ is an automorphism. (β) Let $\tau_1, \cdots, \tau_n$ be a finite set of automorphisms of the multiplicative semigroup of real numbers; then $T_0 : (a_1, a_2, \cdots, a_n, a_{n+1}, \cdots) \rightarrow (\tau_1(a_1), \cdots, \tau_n(a_n), a_{n+1}, \cdots)$ is also an automorphism. (γ) Let $(\rho_1, \rho_2, \cdots)$ be a convergent sequence of positive numbers, $\lim \rho_k \neq 0$, and define the automorphism $E_0^\rho : (a_1, a_2, \cdots) \rightarrow ((\text{sgn}\, a_1) \mid a_1 \mid^{\rho_1}, (\text{sgn}\, a_2) \mid a_2 \mid^{\rho_2}, \cdots)$. We may now state the proposition that to each automorphism σ_0 of C_0 there corresponds an H_0, T_0 and E_0^ρ such that $\sigma_0 = E_0^\rho \cdot T_0 \cdot H_0^*$. This is a consequence of Theorem A, §4, which states that *if X and X' are bicompact spaces and σ is an isomorphism of $C(X)$ on $C(X')$, then there is determined a homeomorphism H of X on X' (see (α)), a finite set of exceptional isolated points $x_1, \cdots, x_n$ and associated automorphisms $\tau_1, \tau_2, \cdots, \tau_n$ of the semigroup of real numbers (see (β)), and a continuous positive function $p(x) \varepsilon C(X)$ (see (γ)) such that for any pair of corresponding functions $\sigma : f \rightarrow f'$ we have $f'(H(x)) = (\text{sgn}\, f(x)) \cdot \mid f(x) \mid^{p(x)}$ for $x \neq x_1, \cdots, x_n$*

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