INTEGRAL GEOMETRY ON SURFACES

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1. **Introduction.** The Integral Geometry on surfaces was initiated by Blaschke and Haimovici [2], [6]. The particular cases of the elliptic and hyperbolic plane were studied by the author in two preceding papers [12], [13].

Starting from the concepts introduced by Blaschke and Haimovici, it is the purpose of this paper to continue the study of the Integral Geometry on a more general class of surfaces.

In §2 we recall the definition of density for sets of geodesics and obtain the expression (2.7) which plays a fundamental role in all subsequent discussions and is used in §3 to obtain some integral formulas related to the set of geodesics which intersect a fixed curve of finite length on the given surface. In §4 we give a definition of convex domains on the surface and apply it to obtain some mean value theorems. In §5 we consider sets whose elements are pairs of geodesics and obtain some integral formulas and a theorem referring to geodesic lines. In §§6 and 7 we are concerned with the definition of cinematic density on surfaces and with the generalization of the Poincaré formula of Integral Geometry [3; 23]. Finally, in §8 we give a very general inequality (8.7) for convex curves on surfaces which contains as a particular case the isoperimetric inequality (8.15) for surfaces of constant gaussian curvature.

We shall restrict ourselves to *complete analytic Riemannian surfaces* in the sense of Hopf and Rinow [9]. Consequently the following two properties are satisfied: (a) every geodesic ray can be continued to infinite length; (b) any pair of points can be joined by a curve of shortest length which is a geodesic. Furthermore we assume throughout the whole paper that our surface satisfies Condition A stated at the end of the Introduction and in §§3, 4 and 5 we add the stronger Condition B stated in §3.

We shall represent the surface by S, and in general it will be useful to use on it geodesic polar coordinates with pole (in each case chosen in a suitable position) represented by O. The first fundamental formula becomes

$$ds^2 = d\rho^2 + g(\rho, \theta)^2 \, d\theta^2. \tag{1.1}$$

The pole O together with the function $g(\rho, \theta)$ is called an *element* of S and will be represented by $E(0, g)$ or simply by E. We suppose that each element E is continuable to the complete surface S in the sense of Rinow [11] and Myers [10]. Then $g(\rho, \theta)$ must be an analytic function of ρ for θ = constant and all $\rho > 0$, which satisfies the conditions

$$g(0, \theta) = 0, \qquad g_\rho(0, \theta) = 1. \tag{1.2}$$

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