HURWITZ SERIES: EISENSTEIN CRITERION

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1. **Introduction.** By an H-series is meant a formal power series of the form

$$f(x) = \sum_{m=0}^{\infty} c_m x^m / m!, \tag{1.1}$$

where the c_m are rational integers (see for example [3; 145]). It is easily verified that the set of series (1.1) constitutes a domain of integrity. The quotient of two H-series is in general not an H-series. If $c_0 = \pm 1$ then the reciprocal of $f(x)$ is again of the same form; such series are accordingly called *unit* series. For the general case it is convenient to define the *order* of $f(x)$ as the integer r such that $c_0 = \cdots = c_{r-1} = 0$, $c_r \neq 0$. We consider in the first place the quotient $g(x)/f(x)$, where $g(x)$ is an H-series of order $\geq$ the order of $f(x)$. We find that the quotient can be written in the form

$$\frac{g(x)}{f(x)} = \frac{1}{c_r} \sum_{m=0}^{\infty} a_m \frac{(r!x)^m}{c_r^m \{m, r\}}, \tag{1.2}$$

where the a_m are rational integers and

$$\{m, r\} = \frac{m!(m+1)! \cdots (m+r)!}{1!2! \cdots r!}, \qquad \{m, 0\} = m!. \tag{1.3}$$

The result (1.2) suggests the consideration of series

$$\sum_{m=0}^{\infty} a_m x^m / \{m, k\}, \tag{1.4}$$

where the a_m are again rational integers. We shall call (1.4) an H-series of *type* k; the totality of series (1.4) will be denoted by $\mathfrak{H}_k$. In particular $\mathfrak{H}_0$ is the set of series (1.1). It is easily proved that the product of two series of type k is again of type k; it follows that $\mathfrak{H}_k$ is a domain of integrity. Also $\mathfrak{H}_0 \subset \mathfrak{H}_k \subset \mathfrak{H}_{k+1}$. Again, generalizing (1.2), the quotient of series of type k is expressible in terms of series of type $k + r$.

We next seek a criterion that a series $w(x) = \sum \alpha_m x^m$ with *rational* coefficients satisfy an equation

$$\sum_{i=0}^{t} A_i(x) w^i = 0, \tag{1.5}$$

where the $A_i(x)$ are H-series. We find that if (1.5) holds then there exist integers $c > 0$, $r \geq 0$ such that

$$c^{m+1}\{m, r\}\alpha_m/(r!)^m \tag{1.6}$$

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