SOME PROPERTIES OF HURWITZ SERIES

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1. **Introduction.** In connection with the coefficients of the lemniscate function, Hurwitz [2] discussed the class of series

$$f(x) = \sum_{m=0}^{\infty} c_m x^m/m! \tag{1.1}$$

where the c_m are rational integers. It is easily verified that the series (1.1) form a domain of integrity. The series with $c_0 = 0$, $c_1 = 1$ are of special interest; in this case the inverse function $\lambda(x) = \sum_1^{\infty} e_m x^m/m!$ also has integral coefficients. In [1] a theorem of the Staudt-Clausen type for the coefficients of $x/f(x)$ was proved as a consequence of the assumption $(m-1)! \mid e_m$.

In the present paper we find it convenient to treat simultaneously the case of rational integral and integral p-adic coefficients c_m. In §§3–5 we prove a number of results for arbitrary series (1.1). In §§6, 7 we use a weakened form of the assumption mentioned above. For example in §6 we show that the assumption $p \mid e_m$ for $m > p$ leads to the congruence $c_{m+p-1} \equiv c_m c_p \pmod{p}$. In §7 we treat the coefficients $\beta_m^{(k)}$ in

$$(x/f(x))^k = \sum_{m=0}^{\infty} \beta_m^{(k)} x^m/m! \qquad (k \geq 1); \tag{1.2}$$

for $f(x) = e^x - 1$, $\beta_m^{(k)}$ reduces to the Bernoulli number $B_m^{(k)}$ in Nörlund's notation [3; Chapter 6]. It is almost immediate that $p^k \beta_m^{(k)}$ is integral $\pmod{p}$; for a sharper result see Theorem 14. In particular for $k = ap^i$, $0 < a < p$, $p\beta_m^{(k)}$ is integral. The following explicit formula may be cited:

$$p\beta_m^{(p^i)} \equiv \begin{cases} -c_p^n \pmod{p} & (m = np^i(p-1)), \\ c_p^n & (m = p^{i-1}(p-1)(np+1)); \end{cases}$$

if m is not of the indicated form then $\beta_m^{(p^i)}$ is integral. For another result of this kind see (7.13).

2. **Notation.** Let $\mathfrak{H}$ denote the set of series (1.1) with integral coefficients and let $\mathfrak{H}(p)$ denote the set of series with integral p-adic coefficients. In particular $\mathfrak{H}(p)$ includes the series with rational c_m such that the denominators are prime to p; also $\mathfrak{H} \subset \mathfrak{H}(p)$ for arbitrary p. If $c_0 = \cdots = c_{r-1} = 0$, $c_r \neq 0$, we shall say that $f(x)$ is of *order* r. If $f(x), g(x) \in \mathfrak{H}$ (or $\mathfrak{H}(p)$) the notation $f(x) \mid g(x)$ indicates that $g(x)/f(x) \in \mathfrak{H}$ (or $\mathfrak{H}(p)$). If $f(x) \in \mathfrak{H}$ and $c_0 = \pm 1$ then $1/f(x)$ is also in $\mathfrak{H}$; similarly if $f(x) \in \mathfrak{H}(p)$ and c_0 is a p-adic unit (that is a p-adic integer

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