A UNIFIED THEORY OF SEMI-CONTINUITY

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1. **Introduction.** Semi-continuity, a concept at one time defined exclusively for real-valued functions of a real variable, has been extended in the past few decades in such a way that the concept is now applicable to functions of a very general nature.

The length function L is an example of a lower semi-continuous function whose domain is not a set of real numbers. The domain of L is a set C of rectifiable curves. The set C is topologized in a suitable way, and for each $c \in C$ we define $L(c)$ to be the length of c.

The period function is a semi-continuous function of a type similar to that of the length function. To obtain this function, let f be a periodic function on a topological space T and for each $x \in T$ define $p(x)$ to be the period of x under f. The function p is a lower semi-continuous integer-valued function on T. Periodic functions and the period function are discussed by Whyburn [5; Chapter XII].

The examples we have given have both been real-valued functions. The upper semi-continuous decompositions introduced by R. L. Moore [4] seem to have been the first semi-continuous functions to have a more general range. Moore did not, however, point out the functional nature of decompositions. It was W. A. Wilson [6] who first formally considered what we shall call set-valued functions. A systematic study of these functions has been made by L. S. Hill [3]. Hill considered semi-continuous functions whose independent variables represented points in Euclidean space and whose dependent variables represented compact subsets of Euclidean space; modern developments make a more general theory desirable.

To obtain examples of semi-continuous set-valued functions, we let f be a continuous function on a compact metric space K into K. If we define $g(x) = f^{-1}f(x)$ for each $x \in K$, then g will be an upper semi-continuous set-valued function on K. If we define $s(x)$ to be the semi-orbit-closure of x under f (see [1]) for each $x \in K$, then s will be a lower semi-continuous set-valued function on K.

We develop in this paper a general theory of continuity which yields as special cases the theories of semi-continuity for both real-valued and set-valued functions. The essence of our theory is the concept of a continuity structure for a metric space.

2. **Continuity structures.** Throughout the remainder of this paper A denotes a topological space, M a metric space having metric ρ, and P the set of all positive real numbers.

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