THE EXTENSION OF HOMEOMORPHISMS

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1. **Introduction.** Let f be a homeomorphism from an open set P in one space to a subset of another. There are occasions when answers to the following questions are of interest. Under what conditions is f extensible continuously to the boundary of P? If f is extensible, then how does the extension behave on the boundary of P? One example of this nature is found in the work of Carathéodory and Osgood on $1 - 1$ conformal maps from the interior of the unit circle to bounded plane regions. Another example is found in the study by Whyburn of the relative distance transformation of Mazurkiewicz as applied to plane regions [8; 154–162].

The purpose of this paper is to study in detail the behavior of the extension, assuming the extensibility of the map. More precisely, let P be an open set of a compact metric space A and let Q be a subset of a compact metric space B. Let f be a continuous map from $\bar{P}$ to $\bar{Q}$ such that $f \mid P$ is a homeomorphism of P onto Q. Our method, then, is as follows: impose conditions on Q alone or on both P and Q; then study the action of the map $f \mid P^*$. (If A is a subset of $\bar{P}$, then we denote by $f \mid A$ the map f restricted to the set A. Also we designate by A^* the set $\bar{A} - A$.)

The second section establishes two general theorems on the action of $f \mid P^*$. In §3, applications are made. In the first place, implications of uniform local connectedness of the regions are studied. For example, if P is uniformly locally connected, then a necessary and sufficient condition that Q be also is that $f \mid P^*$ be monotone. Secondly, the case in which Q is a spherical region is studied. In that case $f \mid P^*$ is non-alternating, generalizing an earlier result of the author [2; 654]. A new characterization of non-alternating maps on boundary curves is also obtained.

In the last section, homology generalizations of the results of the third section on uniform local connectedness are studied. For example, the theorem stated in the preceding paragraph is shown to be true with "uniformly locally connected" and "monotone" replaced by "uniformly locally i-connected, $0 \leq i \leq n$" and "n-monotone" respectively.

2. **General properties of the extension.** Our first result indicates that to obtain interesting results we must either place restrictions on the set Q or on the map f. This will be followed by a series of lemmas and definitions leading up to the two principal theorems of this section.

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