HAAR MEASURE IN UNIFORM STRUCTURES

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1. **Introduction.** In an earlier paper [4] the author showed how the existence and uniqueness theory of Haar measure could be demonstrated in any locally compact metric space satisfying a simple combinatorial congruence axiom. At the end of the paper it was stated that the same methods could be applied to the more general case of a uniform structure. This is the program of the present paper. The setting of the theory is now a uniform structure satisfying the following congruence axiom: the smallest number of x-neighborhoods required to cover a y-neighborhood is finite and the same for all y-neighborhoods. Assuming beyond this only that the neighborhoods are symmetric and in a certain sense densely ordered, we shall deduce the existence of a unique measure which is invariant on the neighborhoods.

As in [2] and [4] the existence and uniqueness of such an invariant measure are obtained simultaneously, thus avoiding the classical paradox of the subject wherein the axiom of choice is needed to pick out an invariant measure and is then found to be theoretically unnecessary since the invariant measure proves to be unique. The fundamental combinatorial lemmas (Lemmas 1–3, §5) upon which the paper rests are essentially contained in [4], but the rest of the paper diverges considerably from [4] in method and content. The sections before §5 are devoted to preliminary material of a general nature, such as the theory of Jordan content (finitely additive measure on "zero-boundary" sets) in uniform structures. The theory of Haar measure proper follows in §§5–8. In the remaining sections various related questions are discussed, such as the nature of the extended measure in the locally-compact completion of the space, and an application yielding a unique measure invariant under a certain kind of group of homeomorphisms of a topological space.

2. **The axioms for** S. Let S be a uniform structure and let $x(p)$ be the x-neighborhood of p. For any set A let $x[A]$ be the sum of the x-neighborhoods intersecting A, and let $A < B$ mean that there exists an index x such that $x[A] \subseteq B$. In terms of these definitions the fact that S is a uniform structure can be stated as follows: the indices form a directed set under a partial ordering, and $x < y$ implies the existence of an index z such that $z[x(p)] \subseteq y(p)$ for all p (abbreviated $z[x] \subseteq y$). The extra assumptions about S necessary for our theory of Haar measure are as follows.

A1. *If $x < y$, there is a z such that $x < z < y$.*

A2. *If $p \in x(q)$ then $q \in x(p)$.*

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