# BANACH ALGEBRAS OF BOUNDED FUNCTIONS

By Bertram Yood

1. **Introduction.** Let $S$ be any set and $S{:}B(S)$ be the Banach algebra of all bounded functions $f(t)$ where for each $t$, $f(t)$ has its values in a commutative $B^*$-algebra $B(t)$ with a unit (see §2 for definitions). In §2 we determine the nature of the maximal ideals of $S : B(S)$. These are in a $1 - 1$ correspondence with pairs of maximal additive ideals in the distributive lattice of subsets of $S$ and certain equivalence classes of elements in the cartesian product of the spaces of maximal ideals of the Banach algebras $B(t)$. The prototype for the algebras $S : B(S)$ is the algebra of all bounded complex-valued functions defined on $S$. For this algebra in the real case (where the present theory is quite the same), the multiplicative linear functionals (or equivalently the maximal ideals) were determined by Šmulian [13].

In §3 we give a generalized Weierstrass theorem for $B^*$-subalgebras of $S : B(S)$. This is obtained as a consequence of the theory of $B^*$-algebras. In §4 we use the theory to obtain a proof of Stone's topological representation for distributive lattices [15]. Finally in §5 we apply these ideas to the theory of two-valued measure functions.

2. **On the maximal ideals of** $S{:}B(S)$. We use the term "Banach algebra" to refer to a vector algebra in which the underlying vector space is a complex Banach space and in which the multiplication satisfies the condition $\| xy \| \leq \| x \| \, \| y \|$. Gelfand [4] whose notation we adopt, in general, used the term "normed ring". Following Rickart [9] we call a Banach algebra a "$B^*$-algebra" if to each element $x$ there corresponds a unique element $x'$, called the adjoint of $x$, with the following properties: (i) $(x')' = x$. (ii) $(xy)' = y'x'$. (iii) If $\alpha$ and $\beta$ are complex numbers and $\alpha^c$ and $\beta^c$ are their complex conjugates, then $(\alpha x + \beta y)' = \alpha^c x' + \beta^c y'$. (iv) $\| x'x \| = \| x \|^2$.

We shall consider only commutative $B^*$-algebras $A$ which possess a unit of norm one. It can be shown that in $A$, $\| x' \| = \| x \|$. Let $\mathfrak{M}$ be the class of all maximal ideals in $A$. Then to each $M$ in $\mathfrak{M}$ there exists a homomorphism of $A$ onto the complex numbers denoted by $x(M)$. If $\mathfrak{M}$ is topologized as in [4], $\mathfrak{M}$ is bicompact. Let $C(\mathfrak{M})$ represent the class of all complex-valued continuous functions defined on $\mathfrak{M}$. Gelfand and Neumark [5; Lemma 1] have shown that $A$ is equivalent to $C(\mathfrak{M})$ under the correspondence $x \leftrightarrow x(M)$. In this correspondence, $x'$ corresponds to the function $x^c(M)$ (see also Arens [1]) and $\| x \| = \sup | x(M) |$, $M \in \mathfrak{M}$. Also $A$ has no radical [4]. By a *multiplicative*

Received December 27, 1947. Presented to the American Mathematical Society, December 29, 1947.