SOME PROPERTIES OF L SETS IN THE PLANE

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It is our purpose to investigate a notion which may be regarded as a generalization of convexity or as a specialization of connectedness. Although many of the concepts developed here extend to higher dimensions, all of the theorems stated are for the plane. The k-dimensional Euclidean space is denoted by $\mathcal{R}_k$, $k \geq 2$.

DEFINITION. A set S in $\mathcal{R}_k$ is called an L_n set if each pair of points in S can be joined by a polygonal line in S having at most n segments.

For simplicity, in $\mathcal{R}_2$ an L_2 set will be called an L set, and $\mathcal{R}_2$ will be designated by $\mathcal{R}$. The theory of L sets seems to be more involved than that of convex sets. For instance, an L set need not be simply connected. We are concerned chiefly with two problems. The first refers to the character of the complements of L sets. This is developed in §1. The second problem is that of characterizing L sets in terms of sets with simple properties. In §2 we show that a simply connected, bounded closed set is an L set if and only if it can be expressed as the sum of convex sets every two of which have a point in common. An interesting illustration of L sets is discussed in the concluding section.

The following notations and definition are used.

Notation. An n-sided polygonal line will be denoted by $P_n \equiv p_1p_2 \cdots p_np_{n+1}$, where the first and last letters correspond to end-points, and where the remaining letters correspond to consecutive vertices. A polygonal line can be degenerate in the sense that any subset of the vertices may be collinear. The infinite line determined by p_i and p_j is designated by $L(p_i, p_j)$, whereas the corresponding line segment is denoted by p_ip_j.

The two-dimensional closed triangle (boundary plus interior) determined by p_i, $i = 1, 2, 3$, is designated by $\Delta p_1p_2p_3$.

DEFINITION. A set S is said to be simply connected if its complement contains no bounded components.

1. Properties of the complement of an L set in $\mathcal{R}$.

The following definition is useful in proving Theorems 1.1 and 1.2.

DEFINITION. A polygonal line P_n lying in a set E with endpoints a and b is said to be irreducible in E if there exists no polygonal line $P_r \subset E$, $r < n$, having a and b as endpoints.

THEOREM 1.1. *If S is a bounded closed L set, then each bounded component of the complement of S is an L set.*

Proof. Suppose C is a bounded component of the complement of S which is

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