CONVERGENCE IN AREA AND CONVERGENCE IN VOLUME

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1. **Introduction.** In a recent paper establishing the spatial isoperimetric inequality in terms of the Lebesgue area, Radó [6] pointed out that the previous attempts of Blaschke [3], Bonnesen [4], and Tonelli [12] made use of the following questionable assumption:

1.1. If a sequence of closed polyhedra converges to a limit surface S and the areas of the polyhedra converge to the Lebesgue area of S, then the volumes of the polyhedra converge to the volume enclosed by S.

Although these earlier attempts do not contain a precise formal definition for enclosed volume, the Besicovitch [2] example (of a simple closed surface which occupies a point set of three-dimensional measure greater than a fixed positive number while its Lebesgue area is arbitrarily small) can be used to show that the above assumption is false if the enclosed volume is interpreted to mean the three-dimensional measure of the set of points interior to the surface or if it is interpreted as the measure of the set of interior points plus the points occupied by the surface (see Radó [6]). Attention is called to the fact that Bonnesen [4] states explicitly that the three-dimensional measure of the point set occupied by the limit surface S should be zero and in that case the Besicovitch example does not apply; however, even in this special case it is not obvious that the assumption 1.1 is true, except when the approximating polyhedra and the limit surface are all simple closed surfaces.

In the present paper we shall use an enclosed volume based on topological considerations and show that for this volume 1.1 is true in the Bonnesen case. As a matter of fact, we can do away with the assumption that the approximating surfaces be polyhedra and hence we have the result that if the limit surface occupies a point set of measure zero, then convergence in Lebesgue area implies convergence in enclosed volume. These statements will be made precise in the next section.

2. **Definitions.** By a *closed surface* we shall mean an oriented F-surface of the type of the 2-sphere (see, for instance, Youngs [13]). If U denotes the positively oriented unit sphere $u^2 + v^2 + w^2 = 1$ in an auxiliary uvw-space, a closed surface S is determined by a representation

$$S: \qquad x = x(p), \qquad y = y(p), \qquad z = z(p) \qquad\qquad (p \in U),$$

where $x(p)$, $y(p)$, $z(p)$ are continuous, real-valued functions on U. The points in Euclidean xyz-space obtained by means of this representation form the point set $[S]$ occupied by the surface. Note that S and $[S]$ are different entities. The

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