THE HOMOLOGY GROUPS OF THE FIBRE BUNDLES OVER A SPHERE

By Hsien-Chung Wang

1. **Introduction.** Let X and B be two topological spaces. We shall call X a *fibre bundle* over B (see [5]) if there exists a mapping π of X onto B having the following properties:

(i) the inverse image $F_p = \pi^{-1}(p)$ of each point p of B is homeomorphic to a fixed space F;

(ii) for each point p of B we can find a neighborhood U_p of p such that there is a homeomorphism ψ carrying the inverse image $\pi^{-1}(U_p)$ onto the topological product $U_p \times F$, and furthermore, $\psi(\pi^{-1}(q)) = q \times F \subset U_p \times F$ for every q of U_p. (Here we adopt the general definition of fibre bundles by taking the group of homeomorphisms of the fibres to be the group of all possible homeomorphisms.)

F_p, π and B are called the *fibre*, the *projection* and the *base space* respectively.

Given spaces B and F, we have not much knowledge about the homology groups of the fibre bundles over B with fibres homeomorphic to F. Gysin [7] and Leray [9] established some inequalities relating the Poincaré polynomials of B, F and the bundle. In this short paper, we shall use the homology sequence to determine the homology groups of the fibre bundles over a sphere. The arguments are quite simple.

As the group of integers is the universal coefficient group for the spaces which concern us, we limit ourselves to integral homology groups only. Let us denote by H^r the r-th integral homology group. Then our main results can be stated as follows.

Theorem 1. *Let X be a fibre bundle over a sphere S^n of n dimensions with fibres homeomorphic to a finite polyhedron F. If $H^s(F) = 0$ for $s \geq n - 1$ (in particular, F is of dimension less than $n - 1$), then*

$$H^r(X) \approx H^r(S^n \times F) \qquad (r = 0, 1, 2, \cdots).$$

Theorem 2. *Let F be a connected finite polyhedron with $H^{n-1}(F) \approx I$, $H^s(F) = 0$, $s > n - 1$ (in particular, F is a closed orientable pseudo-manifold of $n - 1$ dimensions). If X and S^n have the same meaning as in the preceding theorem, then either*

$$H^r(X) \approx H^r(S^n \times F) \qquad (r = 0, 1, 2, \cdots) \tag{1.1}$$

or

$$H^n(X) = 0, \quad H^{n-1}(X) \approx I_m, \quad H^r(X) \approx H^r(S^n \times F) \quad (r \neq n, n - 1). \tag{1.2}$$

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