THE THIRD ITERATE OF THE LAPLACE TRANSFORM

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Introduction. The iterates of the Laplace transform have been the subject of recent studies by Widder [8], Boas and Widder [3], and Pollard [6], [7]. The papers of Widder and Boas and Widder contain very complete results on the second and fourth iterates. The methods and results of the present paper are analogous to those of Boas and Widder [3].

We shall be concerned with triply iterated Laplace integrals, that is, integrals of the form

$$f(x) \equiv \int_{0+}^{\infty} e^{-x\rho}\, d\rho \int_{0+}^{\infty} e^{-\rho\sigma}\, d\sigma \int_{0+}^{\infty} e^{-\sigma\tau}\, d\alpha(\tau), \tag{1}$$

where α is a function of bounded variation in every interval $[\epsilon, R]$, $0 < \epsilon < R < \infty$, and $\int_{0+}^{\infty}$ means $\lim_{\epsilon \to 0, R \to \infty} \int_{\epsilon}^{R}$. α is referred to as *determining function* and f is called *generating function.* By formally changing the order of integration in (1) we obtain the L_3-transform:

$$f(x) = \int_{0+}^{\infty} e^{xt} E(xt)\, d\alpha(t), \tag{2}$$

where $E(x)$ denotes the exponential integral:

$$E(x) = \int_{x}^{\infty} e^{-t} t^{-1}\, dt \qquad (x > 0). \tag{3}$$

Most of our results center about the L_3-transform rather than the triply iterated transform (1). This is justified by the fact that whenever the L_3-transform exists, then the triply iterated transform exists and is equal to the former.

Pollard's work treats all the iterates, but the main result is the inversion of the transforms at points of continuity of the determining function. Our methods are purely real and different from those of Pollard who employs the calculus of residues. Furthermore, the present inversion formula is also established at points of simple discontinuity of the determining function, where it yields the mean value. Chapter III contains a number of representation theorems which seem to be susceptible of generalization to the case of the general iterate. Necessary and sufficient conditions are given which a function must satisfy in order that it be representable as an integral transform with kernel the third iterate of the Laplace kernel, the determining function belonging to certain familiar classes.

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