INEQUALITIES OF THE MARKOFF AND BERNSTEIN TYPE FOR INTEGRAL NORMS

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Introduction. In a paper published in 1889 A. Markoff [7] proved the following theorem, the first of its type to appear in the literature: If $f(x)$ is a polynomial of degree n and $|f(x)| \leq 1$ in the interval $-1 \leq x \leq 1$ then in the same interval $|f'(x)| \leq n^2$. An improvement on this result was obtained in 1912 by S. Bernstein [1] in connection with research on approximations of functions by polynomials. Under the same conditions as above he showed that $|f'(x)| \leq n(1-x^2)^{-\frac{1}{2}}$ where $-1 < x < 1$. Therefore we speak of inequalities which give a bound for the ratio of the norm of the derivative of a polynomial to the norm of the polynomial as inequalities of the Markoff and Bernstein type.

In 1914 M. Riesz [10] incorporated in some related work an inequality of this type. He proved that if $f(z)$ is a polynomial of degree n which on the circle $|z| = 1$ satisfies $|f(z)| \leq 1$, then for z on this circle $|f'(z)| \leq n$. Another extension was given in 1925 by G. Szegö [16] in the following form: If C is a simple closed Jordan curve composed of a finite number of analytic arcs meeting in exterior angles $t_i\pi$, $t_i \neq 0$, and if $f(z)$ is a polynomial of degree n then $|f'(z_0)| < cn^t \max_{z \epsilon C} |f(z)|$, where z_0 is on C, t is the maximum t_i, and c is independent of n and $f(z)$. This result was extended to fractional derivatives by Montel [8] and by Sewell [14].

A notable addition to the list of inequalities of this type was made by Zygmund [20] in 1932 with a result for integral norms instead of absolute values. If $F(\theta)$ is a trigonometric polynomial of degree n and period 2π and if $p \geq 1$, then

$$\left\{(1/2\pi) \int_{-\pi}^{\pi} |F'(\theta)|^p \, d\theta\right\}^{1/p} \leq n\left\{(1/2\pi) \int_{-\pi}^{\pi} |F(\theta)|^p \, d\theta\right\}^{1/p}.$$

This type of inequality was extended to rational polynomials on a straight line segment in 1937 by Hille, Szegö, and Tamarkin [5] in the form

$$\left\{\int_{-1}^{1} |f'(x)|^p \, dx\right\}^{1/p} \leq An^2\left\{\int_{-1}^{1} |f(x)|^p \, dx\right\}^{1/p},$$

where $p \geq 1$ and A is independent of n and $f(x)$.

These are some of the principal steps in the development of the type of inequality under consideration. Their applications in the field of approximations are extensive. A bibliography on the salient points of the theory is to be found in an address by A. C. Schaeffer in 1940 [11].

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