NOTE ON A THEOREM OF CARLITZ

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Some years ago Carlitz [1], [2] investigated the behavior, for $n \to \infty$, of the sum

$$\sum_{n_1 + \cdots + n_s = n} n_1^{\alpha_1} \cdots n_s^{\alpha_s} F_1(n_1) \cdots F_s(n_s),$$

where $F_1, \cdots, F_s$ are arithmetic functions satisfying certain simple conditions, and $\alpha_1, \cdots, \alpha_s$ are non-negative numbers. For this sum he found an asymptotic formula with estimation of remainder. His first treatment of the problem was based on the circle-method of Hardy and Littlewood; later he succeeded in sharpening his original result by means of an entirely elementary argument. My object is to sharpen Carlitz's result still further.

In §§1 and 2 preliminary estimates are derived. In §3 I obtain Carlitz's theorem with an improved error term, and two other theorems of a similar type. Finally, in §4, I consider a few applications of the general theorems.

By suitably modifying the method of the present paper one can study the asymptotic behavior, for $x \to \infty$, of the "conjugate" sum

$$\sum_{n \le x} (n + k_1)^{\alpha_1} \cdots (n + k_s)^{\alpha_s} F_1(n + k_1) \cdots F_s(n + k_s),$$

where $k_1, \cdots, k_s$ are given integers. An account of this problem is to be published later.

Notation. Small Greek letters and the letters x, u denote real numbers; all other small italic letters denote integers, positive unless the contrary is stated; p is reserved for primes; ϵ stands for an arbitrarily small positive number. 0^0 is interpreted as 1.

Throughout the paper σ denotes a fixed positive number, and $\theta = \text{Max}\,(\frac{1}{2}, 1 - \sigma)$. The symbol $n \prec \mathfrak{K}$ means that the integer n belongs to a class $\mathfrak{K}$. The highest common factor of $a_1, \cdots, a_s$ will be denoted by $(a_1, \cdots, a_s)$.

We shall occasionally use the abbreviations

$$\Gamma = \frac{\Gamma(\alpha_1 + 1) \cdots \Gamma(\alpha_s + 1)}{\Gamma(\alpha_1 + \cdots + \alpha_s + s)}, \qquad \alpha = \alpha_1 + \cdots + \alpha_s .$$

It will be convenient to write $a \equiv b\ (\cdot m)$ in place of $a \equiv b \pmod{m}$. We shall also frequently write

$$\sum \{\mathfrak{S}\} f(m, n, \cdots)$$

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