NORMAL PRODUCTS OF MATRICES

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It is known that for any $n \times n$ non-singular matrix A with elements in the complex field there exists a unitary matrix U and two positive definite hermitian matrices H_1 and H_2 such that $A = H_1U = UH_2$ and that this representation is unique [5]; in the singular case a similar representation exists but there is some arbitrariness involved in the representation [2]. For a matrix of order 1×1 this form becomes the familiar $\rho e^{i\theta}$ polar form of a complex number and so this is referred to as the "polar representation" for A. If A is normal (*i.e.*, $AA^* = A^*A$), the hermitian and unitary polar matrices commute, and if a polar representation is had for A for which the polar matrices commute, then A is normal. Also, if two normal matrices commute, their product is a normal matrix; but the converse is not true as a pair of non-commutative unitary matrices shows. The following results are had for normal products of matrices.

Theorem 1. *If A, B and AB are normal matrices, then BA is a normal matrix.*

If A and B commute, the theorem is trivially true. The proof is by induction. The theorem is obviously true for matrices of order 1×1.

Assume the theorem to be true for matrices of order $k \times k$, $k = 1, 2, \cdots, n-1$. Let A, B and AB be $n \times n$ normal matrices; then there exists a unitary matrix U such that $UAU^* = D$ where the diagonal matrix D is in a form such that $d_1d_1^* \geq d_2d_2^* \geq \cdots \geq d_nd_n^*$ where the d_i are the diagonal elements of D, and where d^* is the complex conjugate of d. Let $UBU^* = B_1$ so that $UABU^* = DB_1$ and therefore D, B_1 and DB_1 are normal.

Two possibilities may occur:

(1) $d_1d_1^* = d_jd_j^*$ for all j. Then $D = kD_\mu$ where k is a positive real scalar and D_μ is a unitary diagonal matrix. Therefore,

$$D_\mu^*(DB_1)D_\mu = D_\mu^*(kD_\mu B_1)D_\mu = kB_1D_\mu = B_1kD_\mu = B_1D$$

is normal since it is unitarily similar to a normal matrix. Since B_1D is normal, BA is normal.

(2) $d_1d_1^* \neq d_jd_j^*$ for some j. In this case for some l, $1 \leq l < n$, it is true that $d_1d_1^* = d_2d_2^* = \cdots = d_ld_l^* > d_{l+1}d_{l+1}^* \geq \cdots \geq d_nd_n^*$. Since $B_1 = (b_{rs})$ is normal so that $B_1B_1^* = B_1^*B_1$, the following relations hold:

$$\sum_{i=1}^{n} b_{ji}b_{ji}^* = \sum_{i=1}^{n} b_{ij}b_{ij}^* \qquad (j = 1, 2, \cdots, n); \tag{1}$$

and since $DB_1 = (d_rb_{rs})$ is normal,

$$d_jd_j^* \sum_{i=1}^{n} b_{ji}b_{ji}^* = \sum_{i=1}^{n} d_id_i^*b_{ij}b_{ij}^* \qquad (j = 1, 2, \cdots, n). \tag{2}$$

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