INTEGRAL SETS IN QUASIQUATERNION ALGEBRAS

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Introduction. In 1944, Albert [3] introduced the notion of a quasiquaternion algebra as the simplest of a large class of simple non-associative algebras previously studied by him. These quasiquaternion algebras bore enough resemblance to associative algebras and had simple enough structures to make them seem an ideal testing ground for the generalization of integral theory to non-associative algebras. The present paper, then, is an exploratory attempt to carry out this generalization. We were mainly interested in carrying over the general associative theory (see Jacobson [7; Chapter 6] and Hasse [6]), especially the unique factorization of two-sided ideals. In brief, our conclusions can be stated as follows. For quasiquaternion algebras over any Noether field, integral sets can be defined, *i.e.*, all the generalizations of the definition in the associative case are equivalent (§2). For a large class of quasiquaternion algebras over the rational field the maximal integral sets can be found. Each algebra has one, two or four of them (§3), all of which have approximately the same ideal theory. The behavior of the ideals in such a maximal integral set is completely determined by the ideal theory in the corresponding algebras over all the p-adic fields, as in [6]. For all but a finite number of primes p, the ideals in the p-adic algebra are powers of the (maximal) ideal generated by p so that for these ideals we have the best possible result: they form a commutative semigroup, with divisibility in the semigroup equivalent to inclusion of ideals (§4). For the remaining finite number of primes (those which divide the "discriminant" of the algebra), the multiplication of ideals may be highly irregular (§§4 and 5). Thus there is no hope in general of embedding the integral ideals for a rational algebra in a loop. How much is possible is described in §5. In §6, these results are applied to the related question of valuations. It is found that even over a p-adic field, a valuation of a given quasiquaternion algebra need not exist, though the class of quasiquaternion algebras which do have valuations is by no means small. An explicit statement may be found in Theorem 15 and its corollary.

This paper contains the essential results of the author's doctoral dissertation (Chicago, 1946). Warmest thanks are due to Professor A. A. Albert, under whose guidance this thesis was written.

1. Fundamental properties of quasiquaternion algebras. Given a field F, which throughout this paper is assumed to have characteristic different from two, we construct quasiquaternion algebras over F as follows. Adjoin to F an element u with $u^2 = r$ in F to get the algebra $F(u) = F + Fu$ of order two. Define a linear transformation of period two on $F(u)$ by

$$1 \to 1' = 1, \qquad u \to u' = 1 - u.$$

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