A NORMAL FORM FOR MATRICES WHOSE ELEMENTS ARE HOLOMORPHIC FUNCTIONS

By William G. Leavitt

1. Introduction. A similarity transformation of a matrix A is a transformation of type $T^{-1}AT$, where T^{-1} is the inverse of T. The algebraic theory of similarity transformations of matrices over a field is the subject of considerable literature, but much less is known of the similarity theory of matrices whose elements are functions. Let $A(z) = [a_{ij}(z)]$ be an n-th order matrix each of whose elements is a function of the complex variable z, with characteristic equation

$$(1) \qquad |A - \gamma I| = 0.$$

I is the identity matrix $I = [\delta_{ij}]$ where δ_{ij} are the Kronecker symbols. The present paper is to consider transformations in a closed bounded region R such that, in and on the boundary of R,

(a) Each element of $A(z)$ and each root of (1) is a holomorphic function.

Designate the distinct roots (that is, roots not identically equal over R) of (1) as $\gamma_i(z)$, $i = 1, \cdots, m$, and let their respective multiplicities be h_i, so that $\sum_1^m h_i = n$. By condition (a) the $\gamma_i(z)$ are functions holomorphic over R.

The purpose of this paper is the consideration of the problem of constructing a normal form to which $A(z)$ is reducible by a similarity transformation. The principal result is the proof that there exists a transformation $T(z)$, whose determinant $|T(z)|$ is non-vanishing throughout R, which transforms A into the normal form

$$(2) \qquad U = \begin{bmatrix} \gamma_1 & \phi_{12} & \cdots & \phi_{1n} \\ & \cdot & \cdots & \cdot \\ & & \cdot & \cdot \\ 0 & & & \gamma_m \end{bmatrix},$$

where all the elements under the main diagonal are zero, the ϕ_{ij} holomorphic over R, and each γ_i repeated to its multiplicity h_i.

The set of all functions holomorphic over R will first be shown to satisfy the postulates of a principal ideal ring (§2). It then easily follows from known algebraic theorems that there exists a matrix $S(z)$ of holomorphic functions

Received September 3, 1947; revision received January 16, 1948. Presented to the American Mathematical Society, April 25, 1947. The research for this paper was supported by the Wisconsin Alumni Research Foundation. The author wishes to thank Professor R. E. Langer for his encouragement and assistance in its preparation.