THE GENERALIZED LAPLACE EQUATIONS IN A FUNCTION THEORY FOR COMMUTATIVE ALGEBRAS

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1. **Introduction.** In a paper published in 1893 Scheffers [4] laid the foundation for a theory of analytic functions in linear algebras. He showed that, in a distributive algebra with a principal unit over the complex field, a necessary and sufficient condition that a derivative in the usual sense exist is that the algebra be commutative; and that a necessary and sufficient condition that an integral exist is that the algebra be associative. Hausdorff [1] in 1900 and Ringleb [3] in 1933 extended the function theory to non-commutative algebras. In 1939 Ward [6], using matric methods, succeeded in defining a derivative in the non-commutative case.

In the papers cited the generalized Cauchy-Riemann equations played a prominent role. It is the purpose of this paper to show how the theory associated with the usual Laplace equation can be extended to commutative linear associative algebras.

2. **Analytic functions.** Let $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ be a proper basis for a commutative linear associative algebra A over the real field, where ϵ_1 is the principal unit. Multiplication in A will be defined by

$$\epsilon_i\epsilon_j = \sum_{k=1}^{n} c_{ijk}\epsilon_k \qquad (i, j = 1, 2, \cdots, n), \tag{1}$$

where c_{ijk} are real constants. Let U denote a system of functions $u_i(x_1, x_2, \cdots, x_n)$ of n real variables $x_1, x_2, \cdots, x_n$ where the functions are analytic in a simply-connected region R in n-space. Then

$$u = u_1\epsilon_1 + u_2\epsilon_2 + \cdots + u_n\epsilon_n \qquad (u_i \in U)$$

will be called a function over A of the variable $x = x_1\epsilon_1 + x_2\epsilon_2 + \cdots + x_n\epsilon_n$.

Let $\alpha = a_1\epsilon_1 + a_2\epsilon_2 + \cdots + a_n\epsilon_n$ be a number of A and let R_i denote the matrix (c_{isr}) where r and s are the row indices and column indices, respectively, and the c_{isr} are defined in (1). Since A contains a principal unit, the correspondence

$$\sum_{i=1}^{n} a_i\epsilon_i \leftrightarrow \sum_{i=1}^{n} c_iR_i = R(\alpha)$$

is an isomorphism commonly known as the first regular representation of the algebra A by matrices. Similarly, let $S_i = (c_{ris})$. The correspondence

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