THE ROLE OF THE DIRECTRIX IN LEVI-CIVITA PARALLELISM

By Paul B. Johnson

1. **Introduction.** Whether or not two vectors in an n-dimensional Riemannian space are Levi-Civita parallel depends on the components of the vectors, the space, and on the curve connecting the vectors. We shall examine in detail the role played by this curve, called the directrix. Many authors [2], [3; 65] have discussed the effects of infinitesimal closed directrices, or infinitesimal shifts in a particular directrix. We shall be interested in a theory valid for finite directrices and shifts.

2. **Method.** The finite properties of L-C parallelism are shown to depend on the matrizant functional. Successive Fréchet differentials of the matrizant are computed which lead to a generalized Taylor series expansion. By means of this expansion the effect of any change in the directrix can be examined in detail.

Tensor-matric notation is used throughout. A contravariant vector is written as a column matrix, and a mixed quantity is written as a matric quantity with two less indices. For example, the Christoffel symbols of the second kind in component notation are Γ^r_{ca}. In matric notation they are Γ_a. The usual summation convention is followed. A term in which the same index appears twice, once as a subscript and once as a superscript, represents the sum of n such terms, obtained by giving the repeated index successively the values $1, \cdots, n$.

3. **Integration of the L-C equations.** L-C parallel vectors may be regarded as obtained by displacing a vector parallel to itself along the directrix. Let the directrix have parametric equations $x^i = x^i(\xi)$, $a \leq \xi \leq b$, parameter ξ, $i = 1, \cdots, n$. Let Γ_k be the matrices of the Christoffel symbols of the second kind of the space, and let λ be the column matrix of the contravariant components of the vector being displaced. The matric differential equation of L-C parallel displacement [5] is then

$$d\lambda/d\xi = -\Gamma_k(dx^k/d\xi)\lambda. \tag{3.1}$$

In (3.1) the product $\Gamma_k\lambda$ is the matric product. In component notation $\Gamma_k\lambda = \Gamma^r_{mk}\lambda^m$.

Equations (3.1) may be integrated directly to give

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