THE COHOMOLOGY THEORY OF GROUP EXTENSIONS

By Roger C. Lyndon

1. **Introduction.** A "cohomology theory" of groups has been defined by Eilenberg and MacLane. (The definitions given in §2, as well as the modified forms in §10, are essentially those of S. Eilenberg and S. MacLane [1]. Essentially the same theory, developed in terms of homology rather than cohomology, is given by H. Hopf [5].) The "n-dimensional cohomology groups", $H^n(G, K)$, of an arbitrary abstract group G, into an abelian coefficient group K, are defined in a purely algebraic fashion. These two authors have developed the resulting theory, and have applied it both to algebraic and to topological problems.

We are interested here in the relationship of the cohomology groups of a group G to those of a normal subgroup B of G, and of the corresponding quotient group, $A = G/B$. (Under these circumstances G is said to be an "extension" of B by A. The concept of a *group extension* is discussed in [9] and, in the present connection, by Eilenberg and MacLane [2].) The results obtained depend upon a rather elaborate analysis and are not, in the general case, entirely definitive; for these reasons we shall give a full treatment only for the special case where $G = A \times B$, a direct product, indicating (in §10) what extensions are possible in the general case. In particular, explicit formulas are obtained for the cohomology groups $H^n(G, K)$ whenever both G and K are abelian groups possessing finite bases (and where, in the sense defined below, G operates simply upon K).

It has been recognized that this last result can be accomplished, at least in principle, by a different method. (This is given both by Eilenberg and MacLane [1] and by Hopf [5]; moreover, various extensions are given of the theorem cited here.) This relies upon a topological interpretation for $H^n(G, K)$. Specifically, if G is the fundamental group of an arc-wise connected space X, whose higher homotopy groups up to the n-th all vanish, then $H^n(G, K)$ is isomorphic to the n-th cohomology group of the space X, with coefficients in K. This situation can be realized, for any abelian group G with a finite basis, by taking X as the topological product of a suitable set of toroids and "lens spaces"; hence, by use of Künneth's formulas (see [6] and [7]) for the cohomology groups of a topological product, the group $H^n(G, K)$ can be evaluated.

The treatment in this paper is purely algebraic, and without any recognized adequate topological interpretation.

Received June 16, 1947. The work reported in this paper is taken from a dissertation presented in May, 1946, in partial fulfillment of the requirements for the Ph.D. degree in Mathematics at Harvard University. The work was undertaken under the guidance of Dr. Saunders MacLane, whose advice and encouragement I wish to acknowledge.