THE THEORY OF THE BURKILL INTEGRAL

By Lawrence A. Ringenberg

Introduction. Many results on the differentiation and integration of a function of intervals have appeared in mathematical literature. These results are scattered widely in a literal sense as well as in the sense that there is considerable variance in (i) the space in which the intervals are taken, (ii) the parameter of regularity conditions which are placed on the intervals, (iii) the types of derivatives and integrals, and (iv) the purpose of presenting the results. In this paper we consider functions of intervals in the plane; a fixed parameter of regularity is assumed; and the integrals and derivatives are regular. Our purpose is to coordinate and organize known results and to present several converses and generalizations which, as far as the author is aware, are not in previous literature.

In Chapter I the (Burkill) integrals of a function of intervals are considered. Most of the results state relations which exist between the properties of a function of intervals and the properties of its integrals. One new result is the converse of the following theorem of Saks: If f is a function of intervals of restricted bounded variation, then the upper and lower (indefinite) integrals of f are also functions of restricted bounded variation. The chapter closes with a brief summary of a generalization of a result of Burkill on the extension of the range of definition of the integral of a function of intervals. A complete report of this generalization can be found in [8].

In Chapter II we treat the derivatives of a function of intervals. An important result is the theorem of Saks which states that the upper and lower derivatives of an integrable function of intervals are equal almost everywhere to the upper and lower derivatives of the integral of the function respectively. A considerable portion of this chapter is devoted to lemmas which lead up to this theorem and to various generalizations of it. Another important result concerns functions of type A (type of the Lebesgue area). It may be stated as follows: If f is a function of intervals of type A on an interval I, then the derivative of f exists almost everywhere in I and is summable over I, and the integral of this derivative over I is less than or equal to $f(I)$. Several results of the literature follow as corollaries of this theorem. An important tool which is used frequently in the proofs is the following well-known lemma of Vitali: Let A be a class of intervals, E a measurable set, and $\epsilon > 0$ a real number. If E is covered by the class A in the *sense of Vitali*, that is, if every point in E is contained in a sequence of intervals $I_1, I_2, \cdots$, each of which is an element

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