ALGEBRAIC INDEPENDENCE OF CERTAIN ARITHMETIC FUNCTIONS

By L. I. Wade

The purpose of this note is to give a brief proof of the theorem of R. Bellman and H. N. Shapiro [1] that the five multiplicative arithmetic functions $\sigma(n)$, n, $\phi(n)$, $d(n)$, $2^{\nu(n)}$ are algebraically independent. Here algebraic independence means that the given arithmetic functions do not satisfy, for all n, any polynomial $P(x_1, \cdots) \not\equiv 0$ with real coefficients. We shall use the following lemma whose proof is omitted. *Let $f_1(n), \cdots, f_k(n)$ be k arithmetic functions. If there is a sequence n_i such that $f_j(n_i) \to \infty$ $(n_i \to \infty)$, $j = 1, \cdots, k$, and for every λ, $f_j(n_i)/f_{j+1}^{\lambda}(n_i) \to \infty$ $(n_i \to \infty)$, then $f_1, \cdots, f_k$ are algebraically independent.*

Suppose that $P(x_1, \cdots, x_5) \not\equiv 0$ and $P(\sigma(n), n, \phi(n), d(n), 2^{\nu(n)}) = 0$ for all n. We shall examine the sequence of the form $n = p_1^k \cdots p_k^k pqr$, where $p_1, \cdots, p_k$ are the first k primes, p is a prime $> p_k$, q is the least prime $> p^p$, and r is the least prime $> q^q$. For this sequence $d(n) = 2^3(k+1)^k$, $2^{\nu(n)} = 2^{k+3}$. Since these two functions satisfy the hypotheses of the lemma, we choose k so that $P(x_1, x_2, x_3, d(n), 2^{\nu(n)}) = Q(x_1, x_2, x_3) \not\equiv 0$ and

$$Q(\sigma(n), n, \phi(n)) = 0 \tag{1}$$

on the remainder of the sequence. Now

$$\sigma(n) = \sigma(p_1^k \cdots p_k^k)(p+1)(q+1)(r+1) = a_k(pqr + pq + pr + qr + p + q + r + 1)$$

$$n = p_1^k \cdots p_k^k pqr = b_k pqr,$$

$$\phi(n) = \phi(p_1^k \cdots p_k^k)(p-1)(q-1)(r-1) = c_k(pqr - pq - pr - qr + p + q + r - 1),$$

and as $p \to \infty$,

$$g_1 = \sigma(n) \sim a_k pqr, \qquad g_2 = b_k\sigma(n) - a_k n \sim a_k b_k qr,$$

$$g_3 = b_k c_k \sigma(n) + a_k b_k \phi(n) - 2a_k c_k n \sim 2a_k b_k c_k r,$$

and

$$f_1 = g_3/2a_k b_k c_k \sim r, \qquad f_2 = 2c_k g_2/g_3 \sim q, \qquad f_3 = b_k g_1/g_2 \sim p.$$

However (1) imples that there is an $R(f_1, f_2, f_3) = 0$, $R \not\equiv 0$, on the sequence, but this is a contradiction by the lemma and the choice of q and r.

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Reference

1. Richard Bellman and Harold N. Shapiro, *The algebraic independence of arithmetic functions; (I) Multiplicative functions*, this Journal, vol. 15(1948), pp. 229–235.

Duke University.