THE ALGEBRAIC INDEPENDENCE OF ARITHMETIC FUNCTIONS (I) MULTIPLICATIVE FUNCTIONS

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1. Introduction. The search for relationships between various arithmetic functions has, in the past, resulted in many interesting identities. These efforts have been centered on the problem of evaluating various types of sums of arithmetic functions in terms of others. In this paper we will adopt a negative approach to the problem, and consider the question of when one can assert the algebraic independence of a set of arithmetic functions.

Before proceeding to discuss what types of arithmetic functions we intend to consider, we shall make these terms more precise.

Definition 1. A single-valued function $f(n)$ will be called arithmetic if it is defined to have real values for all positive integral values of n.

Definition 2. An arithmetic function $f(n) \not\equiv 0$ will be called multiplicative if, for m and n relatively prime, $f(mn) = f(m)f(n)$.

Perhaps the best known examples of multiplicative arithmetic functions are n; $d(n)$, the number of divisors of n; $\sigma(n)$ the sum of the divisors of n; $\phi(n)$ the Euler function; $\mu(n)$ the Möbius function; $2^{\nu(n)}$ where $\nu(n)$ is the number of distinct primes dividing n; and the constant 1.

Definition 3. A set of arithmetic functions $f_1, \cdots, f_N$ will be called algebraically independent over the real field if there exists no polynomial $P(x_1, \cdots, x_N) \not\equiv 0$ with real coefficients, irreducible over the real field, such that

$$P(f_1, \cdots, f_N) = 0,$$

for all positive integral values of n.

In this paper we discuss only multiplicative arithmetic functions. We have obtained interesting results for additive functions and general arithmetic functions which we hope to publish subsequently.

An interesting product of the investigations of this paper is the following corollary:

The functions n, $\phi(n)$, $d(n)$, $2^{\nu(n)}$, $\sigma(n)$ and $\mu(n)$ are algebraically independent.

The proof of this result is circuitous in that it seems essential to discuss algebraic relationships between general multiplicative arithmetic functions first.

Since the polynomial relations between two multiplicative functions can be discussed more fully than those involving more than two functions, this case will be discussed first. The extension of some of the results to more than two functions depends upon the results obtained for the case of two.

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