THE NUMBER OF REPRESENTATIONS OF A POLYNOMIAL IN CERTAIN SPECIAL QUADRATIC FORMS

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1. Introduction. The problem of determining the number of representations of a polynomial in $GF[p^n, x]$ in certain special quadratic forms has been considered in several previous papers (see for example [1], [3]). In [1] it was shown that the "singular series" for polynomials [1; (1.3)] furnishes the solution for this problem in a number of cases. However, in all cases considered heretofore, the coefficients of the forms in question were assumed to be elements of $GF(p^n)$. In this paper we shall discuss forms involving polynomial coefficients of the first degree.

More precisely, we consider the following problem. Let B_{ij} represent polynomials of $GF[p^n, x]$ of degree 0 or 1 and let α_i be non-zero elements of $GF(p^n)$ for all values of i. Then we seek the number of solutions of

$$\theta F = \alpha_1 X_1^2 + \cdots + \alpha_m X_m^2 + \sum_{i,j=1}^{s} B_{ij} Y_i Y_j \tag{1.1}$$

in primary polynomials $X_1, \cdots, X_m$ of degree k, and arbitrary polynomials $Y_1, \cdots, Y_s$ of degree less than k, where $\theta = \alpha_1 + \cdots + \alpha_m$ for F primary, $\deg F = 2k$, and θ arbitrary for F of degree less than $2k$. We assume that $m \geq 1$, and further, without loss of generality, that $s \geq 1$.

In §2, it is shown that the singular series furnishes the number of solutions of (1.1) under the assumptions stated above. The singular series for this problem can be written

$$\mathfrak{S}(F; k, t, g) = p^{nk(t-2)} \sum_{\substack{\deg H \leq k \\ H \text{ primary}}} | H |^{-t} \sum_{(G,H)=1} \epsilon(-FG, H) S(gG, H), \tag{1.2}$$

(Note that H is restricted to primary values; this convention will be used throughout the paper.);

$$t = m + s, \tag{1.3}$$

$$g = g(X_1, \cdots, X_m, Y_1, \cdots, Y_s) = \sum_{i=1}^{m} \alpha_i X_i^2 + \sum_{i,j=1}^{s} B_{ij} Y_i Y_j, \tag{1.4}$$

and $S(gG, H)$ is the generalized Gauss sum defined by

$$S(gG, H) = \sum_{X_i, Y_j \pmod H} \epsilon(gG, H) \qquad (i = 1, \cdots, m; j = 1, \cdots, s), \tag{1.5}$$

the X_i, Y_j running independently through complete residue systems (mod H). Notice that for

$$g = \alpha_1 X_1^2 + \cdots + \alpha_t X_t^2,$$

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