THE ASYMPTOTIC NATURE OF SOLUTIONS OF LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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1. We shall prove several theorems on the asymptotic behavior of linear systems of differential equations. We use the notation x for a column vector with components $(x_1, \cdots, x_n)$. All matrices we consider will be square and have complex elements. E is the unit matrix. If A is a matrix then its determinant is $|A|$. A typical result is:

THEOREM 1. *Let*

$$\frac{dx}{dt} = (A + \Phi + R)x, \tag{1.0}$$

where (1) A *is a constant matrix such that* $|A - \mu E| = 0$ *has* n *simple roots* μ_i; (2) Φ *is a matrix such that as* $t \to \infty$, $\Phi \to 0$ *and moreover all the elements of* Φ, $\phi_{ij}(t)$, *satisfy*

$$\int^{\infty} |d\phi_{ij}/dt|\, dt < \infty \qquad (i, j = 1, 2, \cdots, n); \tag{1.1}$$

(3) *the elements of* R, $r_{ij}(t)$, *all satisfy*

$$\int^{\infty} |r_{ij}(t)|\, dt < \infty. \tag{1.2}$$

Φ *and* R *are complex-valued matrices. Let the roots of* $|A + \Phi - \lambda E| = 0$ *be denoted by* $\lambda_i(t)$. *Then* $\lambda_i(t) \to \mu_i$ *as* $t \to \infty$. *Furthermore if the real parts of* μ_i *are not all distinct let* $D_{ij}(t) = \mathrm{Re}\,(\lambda_i(t) - \lambda_j(t))$ *satisfy one of the following three conditions for each* i *and* j:

$$\limsup_{A\to\infty} \left| \int_{t_0}^{A} D_{ij}(t)\, dt \right| < \infty, \tag{1.3}$$

$$\lim_{A\to\infty} \int_{t_0}^{A} D_{ij}(t)\, dt = \infty, \qquad \int_{B}^{A} D_{ij}(t)\, dt > -c \qquad (A > B), \tag{1.4}$$

$$\lim_{A\to\infty} \int_{t_0}^{A} D_{ij}(t)\, dt = -\infty, \qquad \int_{B}^{A} D_{ij}(t)\, dt < c \qquad (A > B), \tag{1.5}$$

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