A TYPE OF INVERSION OF CERTAIN SERIES

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Numerous inversion (or reversion) formulas for finite series have been given, based on various operations such as, among others, the sum and product of classes (as in the principle of cross-classifying classes [14]), Dirichlet multiplication (as in the inversion of Dedekind and Liouville and in one of the theories of numerical functions developed by the writer), and the L. C. M. calculus and its generalizations (as by D. H. Lehmer). The type discussed here concerns relations between the subsets of a given countable set. It was suggested by the inversion formula of H. F. Baker (see [1], confusing misprints of which are rectified in [8; 443–444]), which will be obtained incidentally. (For further references see [8; 441–442], [2], [3], [11], [12].)

1. The sets considered are finite, although under suitable restrictions the formulas can be extended to countably infinite sets. Including X_n and the null set ω, the set X_n of n distinct things, or elements, $x_1, \cdots, x_n$, has 2^n subsets X_t. Each subset X_t containing precisely t of the n elements has the unique complement X'_{n-t} containing the remaining $n - t$ elements of X_n. Each of X_n, X'_{n-t} is the complement of the other. With $X_0 \equiv \omega$, the complement of X_n is ω. For $0 \leq t \leq n$, X_t runs through the ${}_nC_t$ subsets of precisely t elements, where ${}_nC_t$ is the coefficient of p^t in $(1 + p)^n$. With each X_t and its complement X'_{n-t} is associated a symbol $f_t(X_t\,;\,X'_{n-t})$. These symbols are subject to the postulates (1.1) $-$ (1.5).

(1.1) When X_t is assigned, $f_t(X_t\,;\,X'_{n-t})$ is uniquely determinate.

(1.2) If $c, c_t, \cdots$ are any elements of a commutative ring C with zero $0'$ and unity $1'$, $c_tf_t(X_t\,;\,X'_{n-t})$ is uniquely determinate.

(1.3) The set of all $c_tf_t(X_t\,;\,X'_{n-t})$ is a module M.

(1.4) $0'c_tf_t(X_t\,;\,X'_{n-t}) = 0''$, where $0''$ is the zero of M.

(1.5) The elements of M are uniquely determinate.

The addition in M will be denoted by $+$, its inverse by $-$. The same signs will be used presently in connection with operations, *etc.*; the contexts will preclude ambiguity. Likewise the special notations $0'$, $1'$, $0''$ may be replaced by 0, 1, 0 without confusion.

By (1.1), the arrangement of the elements in X_t, X'_{n-t} is immaterial. Since X'_{n-t} is uniquely determined by X_t when X_n is given, it might be suppressed in $f_t(X_t\,;\,X'_{n-t})$. However, sometimes there are advantages in retaining X'_{n-t} in the notation, as may be seen from some of the applications of his formula

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