INVARIANT CHARACTERIZATIONS OF TWO-DIMENSIONAL AFFINE AND METRIC SPACES

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1. **Introduction.** The idea of the algebraic characterizations in invariant form of properties of a space has been presented in a series of papers by T. Y. Thomas [3], [4], [5], [6].

Briefly stated, a property P of a space S is said to have an algebraic characterization if necessary and sufficient conditions for the existence of P can be obtained in the form

$$P_1 = 0, \qquad P_2 \neq 0, \tag{1.1}$$

where P_1 and P_2 are systems of polynomials in the structure functions and their derivatives of S.

In a paper by T. Y. Thomas and the author, [7], several problems were considered whose solutions involved a mixed system

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(u, x), \qquad F_0(u, x) = 0 \qquad (\alpha = 1, \cdots, L), \tag{1.2}$$

$$F_1 = 0, \qquad F_2 = 0, \qquad \cdots, \qquad F_N = 0, \qquad \cdots, \tag{1.3}$$

where ψ_i^α and F_i are linear and homogeneous in the unknowns u.

It was there shown that the vanishing of the resultant system R_L of (1.3) is necessary and sufficient for (1.2) to admit a solution.

It was also shown that the conditions $R_L = 0$ could be expressed by the vanishing of a set of tensors with polynomial components, thus obtaining algebraic characterizations in a simple form.

In this paper the problems of [7] will be considered for $n = 2$, and the tensors whose existence was stated there will be obtained in explicit form.

The problems to be discussed are (1) fields of parallel contravariant vectors, (2) linear first integrals, and (3) metric representation of an affinely connected space.

For problem (3) a slight extension of the definition of algebraic characterization is given. This extension replaces (1.1) by

$$\text{Either } P_0 = 0, \text{ or } P_0 \neq 0,\ P_1 = 0,\ P_2 \neq 0. \tag{1.4}$$

When (1.4) is used it is found possible to give an algebraic characterization of a 2-dimensional metric representation of a 2-dimensional affinely connected space, whereas this is not possible under (1.1). (See [3], [4].)

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