CYCLIC PROJECTIVE PLANES

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1. **Introduction.** Various writers including the author have investigated the representation of projective planes by coordinates from appropriate algebraic systems. (See [4] for an extensive bibliography.) If the group of collineations is sufficiently large, the group itself may yield an alternate representation. Baer [2] has shown the equivalence of certain collineation groups to the Theorem of Desargues.

This paper investigates planes in which the simplest possible group, a cyclic group, of collineations is transitive on the points of the plane, whence the name cyclic projective planes. In §2 the representation given is shown to be a difference set, that is a set of integers (modulo the number of points in the plane for finite planes), such that every integer not zero is a difference of the set in exactly one way. Every cyclic plane is shown to possess a polarity. Section 3 gives a simple method of constructing infinite cyclic planes, and shows that they need not be Desarguesian. Section 4 is concerned with further properties of the finite cyclic planes. In particular Theorem 4.5 shows that they always possess other collineations in addition to the defining cyclic group. These are given by "multipliers" of the difference set. The properties found in this section make it highly plausible that every finite cyclic plane is Desarguesian.

2. **General properties of cyclic planes.**

Definition. A plane π is said to be cyclic with respect to a collineation ϕ if the cyclic group generated by ϕ is transitive on the points of π.

Designating an arbitrary point of π as P_0 the collineation ϕ induces a cycle on the points of π

$$\phi \rightleftarrows (P_0, P_1, \cdots, P_{N-1}), \tag{2.1.1}$$

$$\phi \rightleftarrows (\cdots, P_{-1}, P_0, P_1, \cdots), \tag{2.1.2}$$

where if π is finite (2.1.1) holds with N the number of points in π, and if π is infinite (2.1.2) holds. Hence

$$\phi^j(P_i) = P_t \qquad (t \equiv i + j \pmod{N}). \tag{2.2}$$

In (2.2) and later we shall agree that for the infinite cyclic planes congruences modulo N are to be regarded as equations. As a further convenience in notation a point P_i will be designated by its subscript i alone when no ambiguity can arise. If π is finite then it is well known that $N = n^2 + n + 1$ where $n + 1$ is the number of points on a line.

Received May 14, 1947.