AN EXPOSITION OF THE RELATIVE HOMOTOPY THEORY

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In 1935, Witold Hurewicz defined the homotopy groups of a space and published a number of theorems about them [14]. Later he defined the relative homotopy groups of a space modulo a subset. This definition and a few theorems, contained in a paper written with N. E. Steenrod [15], were published in 1941. Independently, both B. Eckmann [2] and J. H. C. Whitehead [24] introduced the same groups during their investigations on fibre spaces. In a series of lectures at Princeton in 1938, W. Hurewicz proved the analogues for the relative homotopy groups of the theorems announced in his famous four notes [14]. In particular, he established the theorems numbered in the present paper (7.6), (8.3), (12.6), (13.1), (13.3). Since these results have not yet been published, the present author was informed of the situation after he had worked out the demonstrations given in this paper.

The present exposition is to give a detailed treatment of the results of W. Hurewicz mentioned above, some well-known theorems (*e. g.* the homotopy sequence), and a number of published or unpublished works of the present author. Our principal weapon seems to be the homotopy extension property (2.1).

Among the topics completely or almost completely left out we may mention (1) the homotopy relations of fibre spaces in which the original interest of relative homotopy groups lies, [2], [15], [24], (2) the products in relative homotopy groups and their relations with the Whitehead products under the boundary operator ∂, [11], [23], (3) the relative torus homotopy groups of R. H. Fox [9]. Their exclusion is for the sake of simplicity and harmony of our treatment.

1. **Relative homotopy.** Throughout the present exposition, all *spaces* considered are metric and separable, although most of the theorems are true for more general spaces. Without loss of generality [17; 86], we may suppose further that all the spaces are *bounded, i.e.* for a given space X there corresponds a positive number $\Delta(X)$ such that $\rho(x_1, x_2) \leq \Delta(X)$ for each pair of points $x_1, x_2 \in X$. A continuous transformation will be simply called a *mapping*, and a *homotopy* is a continuous family of mappings f_t for $0 \leq t \leq 1$.

Given an abstract set A and two spaces X, Y, let $\{X_\alpha\}$, $\{Y_\alpha\}$ be two systems of sets indexed by A [18; 3] such that $X_\alpha \subset X$ and $Y_\alpha \subset Y$ for each $\alpha \in A$. We shall denote simply by $\{A\}$ the detailed notation $\{X_\alpha, Y_\alpha; \alpha \in A\}$. Con-

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