A VARIATIONAL METHOD IN CONFORMAL MAPPING

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In recent years variational methods have yielded significant results in the study of several classes of schlicht functions. For example, in the case of functions

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots$$

which are regular and schlicht in the unit circle variational methods have, at least in some instances, proved more penetrating than any of the classical methods. It is customary in variational methods to perturb the boundary of the mapped domain in some manner, at the same time investigating the variation in the mapping function which is thus induced. This is often a cumbersome procedure because the boundary of the mapped domain may in some cases be far from "smooth", thereby making it difficult to describe the variation in exact terms. The present paper develops a general variational method which may be used in the interior of the map where the variation is analytic. The results of this paper will be used by the authors in future investigations, and have indeed already been used with only a hint as to the proof in [4].

The classical representation of a closed surface is by means of a plane schlicht region **P**, generally taken to be a polygon, of which the points on any edge are identified with the points on some other edge. The identification is made by means of a one-one continuous mapping in which the point and its image are considered to be identical. If the surface is not closed, some edges remain unidentified and constitute the boundary. In this representation interior points of identified edges are interior points of the surface. If the identification is by means of a one-one analytic mapping, such surfaces have been considered as domains of definition of analytic functions which take the same value at corresponding points of identified edges. A simple example is the fundamental polygon of an automorphic function in which the correspondence between pairs of edges is established by a bilinear transformation. The existence of analytic functions on surfaces which by the boundary correspondence are closed and orientable has been pointed out by Riemann [3; 121] and by Klein [2; 637].

Consider in particular an analytic Jordan arc Γ connecting two finite points α and β of the plane, and suppose that Γ is regular even at its endpoints. Let $p(z)$ be a regular analytic function in some neighborhood containing Γ in its interior, and let $p(\alpha) = p(\beta) = 0$. If ϵ is a sufficiently small complex number, the point $z + \epsilon p(z)$ describes an analytic Jordan arc Γ_ϵ joining α and β as z describes Γ, and we identify the point z of Γ with the point $z + \epsilon p(z)$ of Γ_ϵ.

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