EXTENSION OF ANALYTIC FUNCTIONS

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1. T. Carleman [1] has proved the following theorem.

Theorem A. *If* (a) D_1 *is a domain in the upper half-plane and* D_2 *a domain in the lower half-plane such that their common boundary contains an interval* (a, b) *of the x-axis,* (b) *if* f_i, $i = 1, 2$, *is analytic in* D_i *and* (c) *if*

$$\lim_{y \to 0+} [f_1(x + iy) - f_2(x - iy)] = 0 \tag{1.1}$$

uniformly for $x \in (a, b)$, *then* f_1, f_2 *are parts of the same analytic function* f *which is analytic on* (a, b) *itself.*

The object of this paper is to show how much of the conclusion of this theorem is left if we abandon the uniformity of convergence in (1.1). Theorem B below is also due to Carleman and its Corollary 3.2 shows that boundedness can be substituted for uniformity without affecting the conclusion. Theorem C drops uniformity altogether without introducing any new hypothesis. Theorems D and E introduce majorants of which the last one is able to guarantee the same conclusion as Carleman's theorem. This is in its way a best possible result.

2. *Proof of Carleman's theorem.*

(i) $\phi(x, y) = f_1(x + iy) - f_2(x - iy)$ is a complex-valued harmonic function in $D_1 \frown D_2^*$, where D_2^* denotes the domain in the upper half-plane, conjugate to D_2.

(ii) According to the hypothesis, $\phi(x, y)$ is continuously equal to zero on (a, b) and, by the reflection principle, it can therefore be extended into the lower half-plane by the equation $\phi(x, -y) = -\phi(x, y)$.

(iii) The conjugate harmonic function, defined by the Cauchy-Riemann equations, is

$$\phi^*(x, y) = i^{-1}[f_1(x + iy) + f_2(x - iy)].$$

(iv) The functions $f_1(x + iy) = \phi(x, y) + i\phi^*(x, y)$ and $f_2(x + iy) = \phi(x, -y) - i\phi^*(x, -y)$ are analytic wherever the extended $\phi(x, y)$ is proved to be harmonic.

(v) Hence f_1, f_2 are analytic on (a, b) and, since they assume there the same values, they are parts of the same analytic function f.

3. Theorem B. *If* (a) *of Theorem* A *is satisfied and*

$$\lim_{y \to 0+} \int_\alpha^\beta [f_1(x + iy) - f_2(x - iy)]\, dy = 0 \tag{3.1}$$

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