MULTIPLICATIVE SEQUENCES AND TÖPLERIAN (L^2)-BASES

By Philip Hartman

Let $\phi(t)$ be a periodic function of class (L^2) having the Fourier development

$$(1)\qquad \phi(t) \sim \sum_{m=1}^{\infty} \phi_m \sin mt.$$

Recent papers [8], [2], [9], [3] have dealt with the problem of characterizing those functions $\phi(t)$ for which the sequence of functions

$$(2)\qquad \phi(t), \phi(2t), \cdots$$

is a basis for all functions of class (L^2) on $(0, \pi)$. By an (L^2)-basis on $(0, \pi)$ is meant that there exists, for every function $f(t)$ of class (L^2) on $(0, \pi)$ and for every $\epsilon > 0$, a finite set of constants $c_1, c_2, \cdots, c_n$ (depending on ϵ) such that

$$\int_0^{\pi} | f(t) - \sum_{k=1}^{n} c_k \phi(kt) |^2 \, dt < \epsilon.$$

It may be remarked that the methods and results of the papers cited above are also valid for the situation where (1) is replaced by the even function

$$\phi(t) \sim \sum_{m=1}^{\infty} \phi_m \cos mt \qquad (\phi_0 = 0),$$

and the constant function 1 is adjoined to the sequence (2). The general problem might be treated by using the fact that any periodic function is the sum of an odd and of an even periodic function, all having the same period.

The known criteria for the sequence of functions (2) to be an (L^2)-basis on $(0, \pi)$ are expressed in terms of the Fourier coefficients $\phi_1, \phi_2, \cdots$ and the associated Dirichlet series $\sum \phi_n/n^s$. The coefficients in (1) are uniquely determined by the function $\phi(t)$ and must satisfy

$$(3)\qquad \sum_{m=1}^{\infty} | \phi_m |^2 < \infty$$

(Bessel); conversely, any sequence of numbers $\phi_1, \phi_2, \cdots$ satisfying (3) determines a function (2) uniquely (Fischer-Riesz). A sequence of numbers $\phi_1, \phi_2, \cdots$ is said to be multiplicative if

$$(4)\qquad \phi_1 = 1 \text{ and } \phi_{mn} = \phi_m \phi_n \text{ when } (m, n) = 1,$$

where (m, n) is the greatest common divisor of m and n.

The main theorem (I) will be developed in §1. It involves necessary and sufficient conditions in order that a multiplicative sequence $\phi_1, \phi_2, \cdots$ determine

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