A FIXED POINT THEOREM FOR UPPER SEMI-CONTINUOUS TRANSFORMATIONS ON n-CELLS FOR WHICH THE IMAGES OF POINTS ARE NON-ACYCLIC CONTINUA

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Kakutani [2; 457] has proved a fixed point theorem for a multi-valued upper semi-continuous transformation of an n-cell into a subset of itself for which the image of a point is a convex continuum; and Eilenberg and Montgomery [1; 221] have extended Kakutani's result to transformations for which the images of points are acyclic continua of a very general type. (When the term n-cell is used, reference is made to a closed n-cell.) The results to be established in the present paper are concerned with special types of upper semi-continuous transformations for which the images of points are non-acyclic.

The definition for upper semi-continuous transformations used by Kakutani [2; 457] is:

Definition 1. A multi-valued transformation T of a point set A into a point set B is said to be upper semi-continuous if (a) $P_i \to P_0$, (b) $Q_i \to Q_0$, and (c) Q_i is in $T(P_i)$ for each positive integer i, imply that Q_0 is in $T(P_0)$.

This is equivalent to saying that T is upper semi-continuous if $P_i \to P_0$ implies that $T(P_i) \to E$, where E is a subset of $T(P_0)$, if one understands the notation $T(P_i) \to E$ to mean that the sequence of sets $T(P_i)$ converges sequentially to E.

The following definition will also be of use.

Definition 2. A multi-valued transformation of a point set A into a point set B will be said to be continuous if $P_i \to P_0$ implies that $T(P_i) \to T(P_0)$.

If M is the boundary of a topological n-cell, the bounded and non-bounded complementary domains of M will be called the interior and exterior of M, respectively.

Theorem 1. *If I_n is a topological n-cell, T is a continuous multi-valued transformation of I_n into a subset of itself such that for each point P of I_n, $T(P)$ is the boundary of a topological n-cell and M_1, M_2, and M_3 are the subsets of I_n consisting of the points P which are respectively in the interior of $T(P)$, in $T(P)$, and in the exterior of $T(P)$, then* (a) *M_2 is non-vacuous and closed,* (b) *$M_1 + M_2$ and $M_2 + M_3$ are each closed,* (c) *M_1 and M_3 are each open with respect to I_n; and if M_1 and M_3 are each non-vacuous, then M_2 separates M_1 from M_3 in I_n.*

Proof. Let $S(P)$, for each P in I_n designate the sum of $T(P)$ and its interior. Then since T is upper semi-continuous, S is an upper semi-continuous transformation of I_n into a subset of itself such that $T(P)$, for each P, is acyclic in the sense used by Eilenberg and Montgomery [1; 221], and hence by their theorem, for some point P of I_n, P is in $S(P)$, and therefore is in either $T(P)$

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