A REPRESENTATION OF GREEN'S AND NEUMANN'S FUNCTIONS IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Introduction. Despite the fact that Laplace's differential equation in two variables is a particular case of a partial differential equation of elliptic type, the investigations carried out in this special theory are to a large extent methodologically isolated. The reason is that the powerful tools of complex variables and the theory of analytic functions are not applicable to more general equations.

Recently, new methods have been introduced into the theory of analytic and harmonic functions which are based upon the concept of orthogonal systems, and are of much greater generality than the older ones. They were, in fact, first developed in the theory of analytic functions of two complex variables and led to important results in this difficult branch of analysis [1].

Not long ago it was found possible to make a further advance in this approach and to show that various important functions connected with a plane domain, such as harmonic measure and Green's function, can be expressed in a particularly simple form in terms of orthogonal functions [3] and [10].

In this paper these methods will be developed and applied to the theory of an extended class of partial differential equations of elliptic type. We shall obtain a unified theory and, in particular, a construction for Green's and Neumann's functions for a given domain in terms of orthogonal functions. This construction seems to be of real value for actual computations. Our theory permits further an easy investigation of the variation of the above fundamental solutions with varying domain of definition.

In the last section, we shall discuss in more detail the extent and character of our methods.

1. **Generalities and definitions.** In order to show the ideas in full clarity, without being hampered by too many formalisms, we shall develop the theory at first in the case of a certain simple differential equation. The methods are, however, quite general and in §7 we shall indicate the possible extensions of the theory. Let us consider, therefore, the differential equation

$$\Delta\varphi - P\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - P\varphi = 0, \qquad P(x, y) > 0, \tag{1}$$

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