ABSTRACT TAUBERIAN THEOREMS WITH APPLICATIONS TO POWER SERIES AND HILBERT SERIES

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Introduction. This paper falls into three sections. In the first we establish two abstract Tauberian theorems of the Hardy-Littlewood type. These theorems, although much more general in character than their prototypes, require no new mathematical methods for their proofs, and the reader will recognize everywhere techniques associated with Karamata, Hardy, Littlewood, Wiener, and others. However, the most common forms of Karamata's argument are designed for "one-sided" Tauberian theorems, and these forms of the argument fail in a Banach space, because there is no relation of "order". I therefore give complete proofs. After Theorem 3 we are on quite fresh ground.

It has been pointed out to me that these preliminary results can also be derived from an abstract form of Wiener's Tauberian Theorem given by Hille (see [2; Theorem 2]), but it seems desirable to give a more elementary treatment, avoiding the use of integration in Banach space and the difficult "Closure of Translations Theorem" of Wiener.

The reader who wishes to know more about the classical properties of Hilbert's Double Series, will find an excellent account in Hardy, Littlewood and Pólya [1; 212–213, 226–236], but no such knowledge is required to follow the arguments given here. Certain standard results in function theory have been quoted without proof. Of these, Theorem A is given by Littlewood [3; Theorem 13] while the formulae listed as (C), (D), (E) after the statement of Theorem 7, can be found in any treatise on the gamma function. The formula (F), which is used repeatedly, follows from the identity

$$\binom{a+b+n-1}{n} = \sum_{\mu+\nu=n} \binom{a+\mu-1}{\mu}\binom{b+\nu-1}{\nu}.$$

Lastly, the name "Banach Space" is used for any complete normed linear space, and our conclusions are true both for the real and for the complex varieties.

Theorem 1. *If f is Riemann-integrable in* $(0, 1)$ *and* $a > 0$, *then*

$$(1-x)^a \sum_0^\infty x^n f(x^n)\binom{n+a-1}{n} \to \frac{1}{\Gamma(a)} \int_0^1 f(t)(-\log t)^{a-1}\,dt \quad (x \to 1-0).$$

We shall first prove

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