ABSTRACT THEORY OF INVERSION OF ITERATED SUMMATIONS

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1. **Introduction.** It is known that the Möbius or Dedekind inversion formula has been extended very abstractly by L. Weisner [5] and P. Hall [3]. These writers have not only given directions on how to find the appropriate Möbius μ-function for a general system, but also found some applications to the theory of groups by the formulas they derived.

The main purpose of this paper is to investigate inversion formulas for iterated summations over sets of a general system. Duality and multiplicative property of the m-th Möbius function are discussed.

Weisner defined as "hierarchy" the general system which satisfies six axioms. It was first noticed by Hall that the hierarchy axioms 4 and 5 may be obviated by his enumeration principle [3]. We may therefore define the general system by only four axioms.

Let $\mathfrak{S}$ be any finite system of sub-sets of a given set G which is not empty (G itself is a member of $\mathfrak{S}$). Let the set-theoretic relation $\subseteq$ be defined in $\mathfrak{S}$ and satisfy the following axioms (A, B, X, *etc.*, denote the members of $\mathfrak{S}$).

(1) The relation $\subseteq$ is reflexive: $A \subseteq A$.

(2) The relation $\subseteq$ is asymmetric: $A \subseteq B$ and $B \subseteq A$ imply $A = B$.

(3) The relation $\subseteq$ is transitive: $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$.

(4) For every pair A, B of $\mathfrak{S}$, only a finite number of members X of $\mathfrak{S}$ exist such that $A \subseteq X \subseteq B$.

The system with the relation so defined may be called simply an $\mathfrak{S}$-system. Clearly, there is a large number of important cases satisfying these four axioms, *e.g.* all the sub-groups of a finite group G form an $\mathfrak{S}$-system with respect to the sub-group relation. We shall establish theorems with respect to the general system $\mathfrak{S}$.

2. **Number of chains.** This section is a preparation for the next. We shall now consider the number of chains. If $A \subseteq B$ and $A \neq B$, then we may write $A \subset B$ and say that A is properly contained in B. A system of members $A_0, A_1, \cdots, A_s$ of $\mathfrak{S}$ is called a chain of length s, if $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_s$. Comparatively, $A_0 \subset A_1 \subset \cdots \subset A_t$ is called a proper chain of length t.

Let the number of distinct chains $A \subseteq X_1 \subseteq \cdots \subseteq X_{m-1} \subseteq B$ of length m be denoted by $\tau^{(m)}(A \mid B)$; and the number of proper chains $A \subset X_1 \subset \cdots \subset X_{t-1} \subset B$ of length t by $\lambda^{(t)}(A \mid B)$, where A and B are fixed members of $\mathfrak{S}$. Then (notice that each proper chain of length t ($\leq m$) can produce $\binom{m}{t}$ chains of length m) we have

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