CERTAIN GENERALIZATIONS OF THE WEIERSTRASS APPROXIMATION THEOREM

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The classical approximation theorem of K. Weierstrass [10] states that, given any closed interval $[a, b]$ in the real number system, any continuous real-valued function φ defined for all $x \varepsilon [a, b]$, and any positive real number ϵ, there exists a polynomial $P(x)$ such that $| \varphi(x) - P(x) | < \epsilon$ for all $x \varepsilon [a, b]$. Many generalizations of this celebrated theorem have been proved, some in the direction of limiting the powers of x needed [3], others in the direction of generalizing the type of space on which the continuous functions are to be considered. We shall direct our attention to generalizations of the latter type.

As a matter of convenience, we shall adopt certain notations and definitions, as follows. For any topological space X, we denote the set of all real-valued, continuous, bounded functions defined on X by $\mathfrak{C}^*(X, R)$, and by $\| f \|$ the non-negative number sup $[|f(p)|, p \varepsilon X]$, where f is an arbitrary function in $\mathfrak{C}^*(X, R)$. (Throughout the present paper, a topological space X is taken to mean any set X together with a distinguished family of open subsets $\mathfrak{O}$, such that $\mathfrak{O}$ contains 0 and X and is closed under the formation of arbitrary unions and finite intersections.) A subset $\mathfrak{G}$ of $\mathfrak{C}^*(X, R)$ is said to be a set of analytic generators for $\mathfrak{C}^*(X, R)$ if, for every $\varphi \varepsilon \mathfrak{C}^*(X, R)$ and every positive real number ϵ, there exist functions $f_1, f_2, \cdots, f_k \varepsilon \mathfrak{G}$ and a polynomial (with real coefficients) $P(x_1, x_2, \cdots, x_k)$ in the indeterminates $x_1, x_2, \cdots, x_k$ such that $\| \varphi - P(f_1, f_2, \cdots, f_k) \| < \epsilon$.

M. H. Stone has generalized the original Weierstrass approximation theorem to include, under appropriate hypotheses, all bicompact Hausdorff spaces among the domains on which real-valued bounded continuous functions are to be considered. (See [9; Theorem 82].) His result may be stated in the following way. Let X be any bicompact Hausdorff space, and let $\mathfrak{D}$ be any subset of $\mathfrak{C}^*(X, R)$ such that for every pair p, q of distinct points in X, there exists a function $g \varepsilon \mathfrak{D}$ such that $g(p) \neq g(q)$. ($\mathfrak{G}$ may be said to distinguish between every pair of points of X.) Under these conditions, the set $\mathfrak{D}$ is a set of analytic generators for $\mathfrak{C}^*(X, R)$. Simplified proofs of this Stone-Weierstrass theorem have been given by Kakutani [6], Dunford and Segal [4], and Stone (unpublished). It may be remarked in passing that the theorem fails to be true if X is non-bicompact, as will be proved in Theorem 3 below.

The purpose of the present note is to present certain generalizations of the Stone-Weierstrass theorem which are valid in all completely regular spaces, and of which the Stone-Weierstrass theorem appears as a special case. (A T_0-space X is said to be completely regular if, for every $p \varepsilon X$ and every neighborhood

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