HLAWKA'S THEOREM IN THE GEOMETRY OF NUMBERS

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1. Let K be a convex body in n-dimensional space, of volume V, symmetrical about the origin O. Hlawka's theorem [3], [7], [6] (actually valid under more general conditions) tells us that there exists a lattice, with no point in K other than O, whose determinant does not exceed $V/2\zeta(n)$. Here $\zeta(n) = 1 + 2^{-n} + 3^{-n} + \cdots$, so that the factor $\zeta(n)$ is of little significance when n is large. In a recent paper, Mahler [4] has shown that the number $2\zeta(n)$ can be improved, and that in particular it can be replaced by $3.296 \cdots$. The object of this paper is to obtain some further improvements. Our work was in progress when Dr. Mahler kindly gave us a copy of his paper. Our method is very similar to his, the only essential difference being in our treatment of the case $w > 1$. Our presentation is somewhat different, and as we require Lemma 1 for the proof of Theorem 2, we have preferred to prove Theorem 1 *ab initio*.

Let c_n denote the upper bound of V/Δ for all convex bodies K in n dimensions, where Δ is the determinant of any lattice with no point (except O) in K. We prove

THEOREM 1. (a) *For $n \geq 3$, we have*

$$c_n \geq \frac{2}{n}\left(\frac{c_{n-1}^{n/(n-1)} - 1}{c_{n-1}^{1/(n-1)} - 1}\right). \tag{1}$$

(b) *As $n \to \infty$,*

$$\underline{\lim}\, c_n \geq c,$$

where

$$\log c = 2(1 - 1/c) \qquad (c > 1). \tag{2}$$

The numerical value of c is $4.921 \cdots$. Thus, for large n, there exists a lattice with no point other than O in K, whose determinant is less than $V/4.92$.

When K is a sphere, the result provided by Hlawka's theorem has already been substantially improved by Rogers [6]. By a combination of his method with that used in proving Theorem 1, we obtain a further improvement. The result can be expressed in an arithmetical form as follows. Let Q be any positive definite quadratic form in n variables of determinant 1, and let $M(Q)$ denote the minimum value of Q for integral values of the variables, not all zero. Let γ_n be the least number such that $M(Q) \leq \gamma_n$ for all such Q.

THEOREM 2. *Let $\Theta = 0.596 \cdots$ denote the minimum, for $0 < \eta < \pi$, of*

$$2(\log \pi/\eta)^{-1} \sum_{t=1}^{\infty} e^{-\eta t^2}. \tag{3}$$

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