# THE ASYMPTOTIC DENSITY OF CERTAIN SETS OF REAL NUMBERS

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1. **Introduction.** Before describing our more general results we consider a particular case of special interest. Let $P$ be any (Lebesgue-) measurable set of points on an oriented circle of unit length, and let $p$, $0 \leq p \leq 1$, denote the measure of $P$ and $q = 1 - p$ be the measure of the complementary set. Given a real non-negative number $\alpha$ we say that a point $A$ of the circumference is an *$\alpha$-head* if on every positively oriented arc $AB$ commencing at $A$ and terminating at $B$ the ratio of the measure of points of $P$ to the measure of points of the complementary set is *greater than* $\alpha$. If on every such arc the ratio is *at least* $\alpha$ we say that $A$ is an *$\alpha^{\blacktriangle}$-head.* Our main result may, in this particular case, be stated as follows:

Theorem 1. *For $0 < \alpha \leq p/q$ the sets of $\alpha$-heads and $\alpha^{\blacktriangle}$-heads are measurable and the measure of each of them is $p - q\alpha$.*

It is remarkable that the measure of the set of $\alpha$-heads (or $\alpha^{\blacktriangle}$-heads) is determined by the measure of $P$ and is entirely independent of any other properties of $P$, *i.e.* of the distribution of its points on the circumference. (The set of $\alpha^{\blacktriangle}$-heads is closed (see §2) and thus measurable; on the other hand the measurability of the set of $\alpha$-heads necessitates a proof (except in the most simple cases *e.g.* if $P$ consists of a finite number of arcs, when our entire argument can be greatly simplified).) This contrasts sharply with the situation in the corresponding discrete problem. This problem is treated in [1]. There $P$ consists of $p$ white and the complementary set of $q$ black balls, and exact bounds for the numbers $N(\alpha)$ ($N(\alpha^{\blacktriangle})$) of $\alpha$-heads ($\alpha^{\blacktriangle}$-heads) for given $\alpha$, $p$, $q$ are obtained. In general—the most notable exception for $\alpha$-heads occurring when $\alpha$ is an integer—these numbers depend to a certain extent on the distribution of the white and black balls around the circumference. Moreover, when $p, q \to \infty$ while $p/q$ tends to a finite limit $\lambda > \alpha$, the difference between max $N(\alpha^{\blacktriangle})$ and min $N(\alpha^{\blacktriangle})$ is not $o(p)$. (For $\alpha$-heads there is an exception for integral $\alpha$ when max $N(\alpha)$ = min $N(\alpha)$.) Since the discrete problem is closely related to the ballot problem in the theory of probabilities, we restate Theorem 1 in the language of that theory:

*If $0 < \alpha \leq p/q$ then the probability of a point chosen at random on the circumference being an $\alpha$-head (or an $\alpha^{\blacktriangle}$-head) is $p - q\alpha$.*

Instead of regarding $P$ as a set of points on the circumference we could equally well look upon it as a set of real numbers having the property that whenever $x$ belongs to $P$, so do all numbers $x + n$ ($n = \pm 1, \pm 2, \cdots$).

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