SMALLEST FIELDS OVER WHICH EVERY POLYNOMIAL IS SOLVABLE BY RADICALS

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Introduction. A polynomial $f(x)$ in a field K_0 is said to be *solvable by radicals* over K_0 if there exists a chain of fields $K_0 \subset K_1 \subset \cdots \subset K_s$, $K_s \supseteq W_f$, where K_i is pure and of prime degree over K_{i-1} $(i = 1, 2, \cdots, s)$ and W_f is the root field of $f(x)$ over K_0 [2; Def. 40]. A field over which every polynomial is solvable by radicals will be said to have the property S_R and we shall be concerned here with the existence of fields irreducible with respect to this property. Such fields must be absolutely algebraic (algebraic over the prime field) since the maximal absolutely algebraic subfield of a field having the property S_R likewise has this property. We shall see that there exists a smallest field of prime characteristic p having the property S_R, but though every field of characteristic zero having the property S_R has at least one subfield irreducible with respect to this property, these fields are not all pair-wise isomorphic.

1. **The fields of prime characteristic.** An absolutely algebraic field of prime characteristic is uniquely defined by its characteristic and absolute degree [9; 79–88]. We shall denote by $A_{p,m}$ the absolutely algebraic field of prime characteristic p and absolute degree m. If $m = \Pi q_i^{r_i}$ and $r_i = 0$ for every $q_i \neq p$, and $r_i = \infty$ for $q_i = p$, we write $m = p^\infty$ and have

Theorem 1. *The A_{p,p^∞} is the smallest field of prime characteristic p having the property S_R, i.e., the A_{p,p^∞} has the following properties:*

1. *The A_{p,p^∞} has the property S_R.*
2. *The A_{p,p^∞} is a subfield of every field of prime characteristic p having the the property S_R.*

Proof. A necessary and sufficient condition that every cyclotomic polynomial in the $A_{p,m}$, whose roots are the $\varphi(n)$ distinct primitive n-th roots of unity, be solvable by radicals over the $A_{p,m}$ for every $n \not\equiv 0 \pmod p$ is that m be divisible by p^∞ [1; Theorem 8]. Since every finite algebraic extension of the $A_{p,m}$ is cyclotomic, this is, moreover, a necessary and sufficient condition that the $A_{p,m}$ have the property S_R. Hence in the continuum of absolutely algebraic fields $A_{p,m}$ of prime characteristic p having the property S_R obtained by letting the exponents $x_i (i \neq k)$ in $m = q_1^{x_1} q_2^{x_2} \cdots q_k^{x_k} \cdots$ (where $q_k = p$ and $x_k = \infty$) vary independently over the non-negative integers and the symbol ∞, the A_{p,p^∞} is the smallest field having the property S_R. Since any field of prime characteristic p having the property S_R must contain as its maximal absolutely algebraic subfield one of the above continuum of fields, the A_{p,p^∞} is, moreover, the smallest field of prime characteristic p having the property S_R.

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