THE MINIMUM MODULUS OF A BOUNDED ANALYTIC FUNCTION

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1. **Introduction.** A study of several simple examples will suffice to show that a good deal of diversity can be expected in the behavior of the minimum modulus of an analytic function. In the case of a function possessing zeros, the minimum modulus cannot be expected to mimic in the large the maximum modulus with respect to monotoneity and convexity properties, and this fact contributes in no small part to the difficulty of the study of the minimum modulus. For certain classes of entire functions information is available on the behavior of the minimum modulus [3], [10]. For entire functions in general little is known. In the present paper we shall be concerned with the study of the minimum modulus of bounded analytic functions. More explicitly, let $f(z)$ denote a function which is defined, analytic and of modulus less than one for $|z| < 1$. We consider the *minimum modulus* of $f(z)$,

$$m(r; f) = \min_{\{\theta\}} |f(re^{i\theta})| \qquad (r, \theta \text{ real}, 0 \leq r < 1) \tag{1.1}$$

and in particular its behavior as r tends to one. (When clear in its context, the notation $m(r)$ will replace $m(r; f)$. This remark applies to all related quantities.)

Along with the minimum modulus of $f(z)$, we shall also consider the function $m^*(r; f)$ defined by

$$m^*(r; f) = \underset{r \leq \rho < 1}{\text{l.u.b.}}\ m(\rho; f); \tag{1.2}$$

the function $m^*(r, S; f)$ where S is a hyperbolic linear fractional transformation with fixed points 1 and -1,

$$S : \frac{Z - 1}{Z + 1} = \lambda \frac{z - 1}{z + 1} \qquad (\lambda \text{ real}, 0 < \lambda < 1),$$

and $m^*(r, S; f)$ is defined by

$$m^*(r, S; f) = \max_{r \leq \rho \leq Sr} m(\rho; f); \tag{1.3}$$

that is, $m^*(r, S; f)$ is the maximum of the minimum modulus of f on the interval of given non-euclidean length whose endpoint nearer to 0 is r. Finally we shall consider functions $\Phi(r)$ which are defined, real-valued, positive, continuous, and non-increasing for $(0 \leq r < 1)$ as candidate majorants of the minimum modulus, as well as certain related functions (see §8 and §9). It will be convenient to use the notation $M(r; f)$ for $\max_{\{\theta\}} |f(re^{i\theta})|$. In §4 certain general results will be given concerning the behavior of

$$\liminf_{r \to 1} m(r; f), \qquad \limsup_{r \to 1} m(r; f).$$

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