THE INVERSION OF THE TRANSFORMS WITH REITERATED STIELTJES KERNELS

By Harry Pollard

1. **Introduction.** Let $H_1(x, y)$ denote the Stieltjes kernel $(x + y)^{-1}$. Then the iterated kernels $H_n(x, y)$ are defined by

$$H_n(x, y) = \int_0^\infty \frac{H_{n-1}(x, u)}{u + y} \, du \qquad (n = 2, 3, \cdots). \tag{1.0}$$

We shall be concerned with the inversion of the S_n-transform defined by

$$f(x) = \int_{0+}^\infty H_n(x, y) \, d\alpha(y) = \lim_{\substack{R \to \infty \\ \epsilon \to 0+}} \int_\epsilon^R H_n(x, y) \, d\alpha(y), \tag{1.1}$$

where $\alpha(y)$ is of bounded variation in every interval (ϵ, R). An inversion formula, depending on the values of $f(x)$ in the complex domain, has been obtained recently by R. Laguardia and J. Lifschitz; their results are as yet unpublished. We shall obtain a real inversion formula by means of a linear differential operator like that used by Widder [6] to treat the case $n = 1$, and by Boas and Widder [1] for the case $n = 2$.

Let us define the operator $L_{k,x}[f]$, $k = 0, 1, 2, \cdots$, by the formula

$$L_{k,x}[f] = \left\{\Gamma\left(k + \frac{1}{2}\right)\right\}^{-2} (-x)^k [x^k f(x)]^{(2k)}. \tag{1.2}$$

$L^n_{k,x}[f]$ will denote the result of n successive applications of this operator to $f(x)$. Then it will be established that (1.1) is inverted by

$$\lim_{k \to \infty} \int_{0+}^x L^n_{k,t}[f] \, dt = \alpha(x) - \alpha(0+) \tag{1.3}$$

at all points of continuity of $\alpha(x)$. For $n = 1$, $n = 2$ this solution is essentially that of the writers just mentioned. Our method is, however, entirely different even in these special cases.

With the exception of the last section this entire paper will be concerned with a proof of (1.3). In §10 it will be shown that the inversion problem in the complex domain can be reduced to the case $n = 1$ by a simple device.

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