A LOCAL MAXIMUM PROPERTY OF THE FOURTH COEFFICIENT OF SCHLICHT FUNCTIONS

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1. Let the power series

$$w = f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots \tag{1}$$

converge for z in the circle of radius 1 with the center at the origin, that is, for $|z| < 1$. If to any two distinct values of z in this circle the corresponding values of w are also distinct, the power series defines a one-to-one conformal mapping of the unit circle $|z| < 1$ onto some region in the w-plane. As z varies throughout the unit circle the corresponding point w varies over the region in the w-plane covering each point of the region once and only once. The property that the power series defines a one-to-one mapping of the unit circle onto a region in the w-plane restricts the coefficients $a_2, a_3, \cdots$ in the power series, and the problem of determining what values these coefficients may have is an extremely difficult one. The coefficients $a_2, a_3, \cdots$ are in general complex numbers, and the so-called coefficient problem in the theory of schlicht functions may be stated in the following form: What is the region V_n of points $(a_2, a_3, \cdots, a_n)$ in $2n - 2$ dimensional real euclidean space which correspond to schlicht power series of the form (1)? As is well known, Faber and Bieberbach proved that $|a_2| \leq 2$. Thus V_2 is a circle of radius 2. Löwner [2] showed that $|a_3| \leq 3$. Schaeffer and Spencer [3] gave a second proof that $|a_3| \leq 3$, and they have also found a method which yields the regions V_n of variability of the coefficients: implicitly for $n > 3$ and explicitly in terms of elementary functions in the case $n = 3$. (See [4].) In the present paper it is shown that $|a_4|$ has a local maximum for the function $f(z) = z/(1 - e^{i\phi}z)^2 = z + 2e^{i\phi}z^2 + 3e^{2i\phi}z^3 + 4e^{3i\phi}z^4 + \cdots$, thus completing certain calculations indicated by Joh [1]. More precisely, we shall prove the following theorem:

If Löwner's k-function has the form

$$k(t) = e^{i[\alpha_0 + \alpha(t)]}, \tag{1a}$$

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