

SETS OF FINITE PLANAR ORDER

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Introduction. Consider a euclidean space of dimension ν ; any euclidean subspace of dimension $\nu - 1$ in this space will be called a *plane*. Let E be a set in the space and let τ be a finite cardinal number. If every plane intersects E in at most τ points, then E is said to be of *planar order* τ . If every plane which intersects E intersects E in exactly τ points, then E is said to be of *exact planar order* τ . If the set E is of planar order $\tau = \nu$, we shall say that E is of *minimal planar order*.

The first section of this paper is concerned with connected sets of finite planar order. Most of the results which have previously been obtained about such sets require in addition that the sets be compact. We make no assumptions about the compactness or closedness of these sets; in fact, we prove that the closure of a connected set of finite planar order is also of finite planar order. The second section, and greater part of the paper, is devoted to the discussion of connected sets of minimal planar order. We show that any connected set of minimal planar order is a simple continuous curve lying in the boundary of a convex set. Finally, in the third section, we prove that a set of exact minimal planar order is totally disconnected and is not the sum of countably many closed sets.

Most of the results in this paper are proved by a combination of synthetic geometric and topological methods.

Preliminary Considerations. We shall be concerned only with sets of planar order $\tau \geq \nu$. The reason for this is that a set of planar order $\tau < \nu$ contains at most τ points and consequently is of little interest. We prove that a set E of planar order $\tau < \nu$ contains at most τ points as follows. Suppose otherwise. Then some subset of E contains more than τ points and at most ν points. Since it contains at most ν points, it lies in a plane; and since it contains more than τ points, it is not of planar order τ . This is a contradiction. Thus the lowest planar order of interest is the planar order $\tau = \nu$, that is, minimal planar order.

Now the lowest dimension of interest is the dimension $\nu = 2$, because every set in a euclidean space of dimension $\nu = 1$ is of exact minimal planar order. Accordingly we shall hereafter suppose that $\tau \geq \nu \geq 2$.

Unless otherwise noted a point set will be assumed to be a subset of a euclidean space. The dimension of this space will always be designated by ν . Insofar as possible we will denote point sets by capital letters, points by lower case italic letters, integers by lower case Greek letters, and real numbers by either Greek or italic lower case letters. Although we employ the same symbols for

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