BOUNDED ANALYTIC FUNCTIONS

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1. Painlevé's problem.

1.1. Some of the most important properties of single-valued analytic functions $f(z)$ can be derived from the simple fact that $\log |f(z)|$ is a harmonic function, except for logarithmic poles. This method of proof has, however, a serious limitation: it is not true conversely that every harmonic function is the logarithm of the absolute value of a single-valued analytic function. In fact, this will be the case only if the periods of the conjugate function are integer multiples of 2π, and without taking this diophantine condition into account it is impossible to obtain best-possible results pertaining to the class of single-valued analytic functions.

The removal of this systematic error, which presents itself as soon as we consider multiply-connected regions, is undoubtedly one of the most important and most difficult problems in the geometric theory of functions. To date very few attempts have been made in this direction, the most remarkable achievement being O. Teichmüller's and M. Heins's treatment of Hadamard's three circles theorem. (After I had completed this paper, my attention was called to two papers by H. Grunsky, *Eindeutige beschränkte Funktionen in mehrfach zusammenhängenden Gebieten I-II*, Jahresberichte der deutschen Mathematikervereinigung, vol. 50(1940) and vol. 52(1942), where the author attacks some problems closely related to the subject of my own paper. His results, though interesting, are far from complete.)

In this paper we shall be mainly concerned with the true equivalent of Schwarz's lemma for single-valued functions in a multiply-connected domain. This question has definite bearing on a classical problem of Painlevé, and we shall begin with some remarks on that problem.

1.2. Painlevé's problem can be stated as follows:

Let E be a compact set in the complex plane. Under what conditions does there exist a non-constant function $f(z)$ which is single-valued, analytic and bounded outside of E?

It may be noted that the corresponding problem for harmonic functions has long been solved. The necessary and sufficient condition for the existence of a non-constant bounded harmonic function outside of E is that E be of positive logarithmic capacity. Naturally, this condition is then also necessary for our present problem, but by no means sufficient.

Two elementary conditions, which are reasonably close to each other, can be derived almost at once. Suppose, in the first place, that E has positive New-

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